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LOG-UNIRULED AFFINE VARIETIES WITHOUT CYLINDER-LIKE OPEN SUBSETS

by Adrien Dubouloz & Takashi Kishimoto

Abstract. — A classical result of Miyanishi-Sugie and Keel-M^{c}Kernan asserts that for smooth affine surfaces, $\mathbb{A}^1$-uniruledness is equivalent to $\mathbb{A}^1$-ruledness, both properties being in fact equivalent to the negativity of the logarithmic Kodaira dimension. Here we show in contrast that starting from dimension three, there exist smooth affine varieties which are $\mathbb{A}^1$-uniruled but not $\mathbb{A}^1$-ruled.

Résumé (*Variétés affines log-uniréglées ne contenant pas d'ouverts cylindriques*)
D'après une caractérisation due à Miyanishi-Sugie et Keel-McKernan, une surface affine lisse S est $\mathbb{A}^1$-uniréglée si et seulement si elle est $\mathbb{A}^1$-réglée, ces deux propriétés étant en fait équivalentes à la négativité de la dimension de Kodaira logarithmique de S. Nous montrons dans cet article que cette équivalence ne subsiste pas en dimension supérieure ou égale à trois.

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Introduction

Complex uniruled projective varieties are nowadays considered through the Minimal Model Program (MMP) as the natural generalization to higher dimensions of birationally ruled surfaces (see e.g., [14]). In particular, as in the case of ruled surfaces, these are the varieties for which the program does not yield a minimal model, but a Mori fiber space. These varieties are also conjectured to be the natural generalization to higher dimensions of surfaces of negative Kodaira dimension, the conjecture being in fact established as long as the abundance conjecture holds true [13], hence in particular for smooth threefolds. During the past decades, the systematic study of the geometry of rational curves on these varieties has been the source of many progress in the structure theory for higher dimensional, possibly singular, projective varieties to which the MMP can be applied. The situation is much less clear for non complete varieties, in particular for affine ones.

The natural analogue of ruledness in this context is the notion of $\mathbb{A}^1$-ruledness, a variety X being called $\mathbb{A}^1$-*ruled* if it contains a Zariski dense open subset U of the form $U \simeq \mathbb{A}^1 \times Y$ for a suitable quasi-projective variety Y. A landmark result of Miyanishi-Sugie [16] asserts that a smooth affine surface is $\mathbb{A}^1$-ruled if and only if it has negative logarithmic Kodaira dimension (see [7] for the definition). For such surfaces, the projection $\mathrm{pr}_Y : U \simeq \mathbb{A}^1 \times Y \to Y$ always extends to a fibration $p : X \to C$ with general fibers isomorphic to the affine line $\mathbb{A}^1$ over an open subset C of a smooth projective model of Y, providing the affine counterpart of the fact that a smooth birationally ruled projective surface has the structure of a fibration with general fibers isomorphic to $\mathbb{P}^1$ over a smooth projective curve. This result, together with the description of the geometry of degenerate fibers of these fibrations, has been one of the cornerstones of the structure theory of smooth affine surfaces developed during the past decades. But in contrast, the foundations for a systematic study of $\mathbb{A}^1$-ruled affine threefolds have been only laid recently in [6].

On the other hand, from the point of view of logarithmic Kodaira dimension, the appropriate counterpart of the notion of uniruledness for a non necessarily complete variety X is to require that X is generically covered by images of the affine line $\mathbb{A}^1$, in the sense that the set of points $x \in X$ with the property that there exists a non constant morphism $f_x : \mathbb{A}^1 \to X$ such that $x \in f_x(\mathbb{A}^1)$ is dense in X with respect to the Zariski topology. Such varieties are called $\mathbb{A}^1$-*uniruled*, or *log-uniruled* after Keel and M^c^Kernan [10], and can be equivalently characterized by the property that they admit an open embedding into a complete variety $\overline{X}$ which is covered by proper rational curves meeting the boundary $\overline{X} \setminus X$ in at most one point. In particular, a smooth $\mathbb{A}^1$-uniruled quasi-projective variety X has negative logarithmic Kodaira dimension. It is conjectured that the converse holds true in any dimension, but so far, the conjecture has been only established in the case of surfaces by Keel and McKernan [10].

It follows in particular from these results that for smooth affine surfaces the notions of $\mathbb{A}^1$-ruledness and $\mathbb{A}^1$-uniruledness coincide. Pursuing further the analogy with the classical projective notions, it seems then natural to expect that these do no longer coincide for higher dimensional affine varieties. Our main result confirms that this is indeed the case. Namely, we establish the following:

THEOREM. — *For every $n \geq 3$, the complement of a smooth hypersurface Z_n of degree n in $\mathbb{P}^n$ is $\mathbb{A}^1$-uniruled but not $\mathbb{A}^1$-ruled.*

The anti-ampleness of the divisor $K_{\mathbb{P}^n} + Z_n$ enables to easily deduce the $\mathbb{A}^1$-uniruledness of affine varieties of the form $\mathbb{P}^n \setminus Z_n$ from the general log-deformation theory for rational curves developed by Keel and McKernan [10]. The failure of $\mathbb{A}^1$-ruledness is then obtained in a more indirect fashion. Indeed, it turns out that for the varieties under consideration, $\mathbb{A}^1$-ruledness is equivalent to the stronger property that they admit a non trivial action of the additive group $\mathbb{G}_a$. We then exploit two deep results of projective geometry, namely the non rationality of the smooth cubic threefold in $\mathbb{P}^4$ in the case $n = 3$ and the birational super-rigidity of smooth hypersurfaces $Z_n \subset \mathbb{P}^n$ if $n \geq 4$, to exclude the existence of such non trivial actions.

In the last section, we consider more closely the case of complements of smooth cubic surfaces in $\mathbb{P}^3$ which provides a good illustration of the subtle but crucial difference between the two notions of $\mathbb{A}^1$-ruledness and $\mathbb{A}^1$-uniruledness in higher dimension. We show in particular that even though such complements are not $\mathbb{A}^1$-ruled they admit natural fibrations by $\mathbb{A}^1$-ruled affine surfaces. We study automorphisms of such fibrations in relation with the problem of deciding whether every automorphism of the complement of a smooth cubic surface is induced by a linear transformation of the ambient space $\mathbb{P}^3$.

1. Recollection on affine ruled and affine uniruled varieties

1.1. Affine ruledness and algebraic $\mathbb{G}_a$-actions. — Here we review general properties of affine ruled varieties, with a particular focus on the interplay between $\mathbb{A}^1$-ruledness of an affine variety and the existence of non trivial algebraic actions of the additive group $\mathbb{G}_a$ on it. We refer the reader to [5] for basic properties of the correspondence between such actions on affine varieties and their algebraic counterpart, the so-called locally nilpotent derivations of their coordinate rings.