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RATIONAL BLANCHFIELD FORMS, S-EQUIVALENCE, AND NULL LP-SURGERIES

by Delphine Moussard

Abstract. — Null Lagrangian-preserving surgeries are a generalization of the Garoufalidis and Rozansky null-moves, that these authors introduced to study the Kricker lift of the Kontsevich integral, in the setting of pairs (M, K) composed of a rational homology 3-sphere M and a null-homologous knot K in M. They are defined as replacements of null-homologous rational homology handlebodies of $M \setminus K$ by other such handlebodies with identical Lagrangian. A null Lagrangian-preserving surgery induces a canonical isomorphism between the Alexander $\mathbb{Q}[t^{\pm 1}]$-modules of the involved pairs, which preserves the Blanchfield form. Conversely, we prove that a fixed isomorphism between Alexander $\mathbb{Q}[t^{\pm 1}]$-modules which preserves the Blanchfield form can be realized, up to multiplication by a power of t, by a finite sequence of null Lagrangian-preserving surgeries. We also prove that such classes of isomorphisms can be realized by rational S-equivalences. In the case of integral homology spheres, we prove similar realization results for a fixed isomorphism between Alexander $\mathbb{Z}[t^{\pm 1}]$-modules.

Résumé (*Formes de Blanchfield rationnelles, S-équivalence, et chirurgies LP nulles*)
Les chirurgies LP nulles sont une généralisation du “null-move” de Garoufalidis et Rozansky, que ces auteurs ont introduit pour étudier le relèvement de Kricker de l’intégrale de Kontsevich, dans le cadre des paires (M, K) composées d’une sphère d’homologie rationnelle M et d’un nœud homologiquement trivial K dans M. Elles sont définies comme des remplacements de corps en anses d’homologie rationnelle homologiquement triviaux dans $M \setminus K$ par d’autres tels corps en anses de même lagrangien. Une chirurgie LP nulle induit un isomorphisme canonique entre les $\mathbb{Q}[t^{\pm 1}]$-modules

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d'Alexander des paires concernées, qui préserve la forme de Blanchfield. Réciproquement, on prouve qu'un isomorphisme fixé entre des $\mathbb{Q}[t^{\pm 1}]$-modules d'Alexander, qui préserve la forme de Blanchfield, peut être réalisé, à multiplication près par une puissance de t, par une suite finie de chirurgies LP nulles. On montre aussi que ces classes d'isomorphismes peuvent être réalisées par S-équivalence rationnelle. Dans le cas des sphères d'homologie entière, on prouve des résultats de réalisation similaires pour un isomorphisme fixé entre des $\mathbb{Z}[t^{\pm 1}]$-modules d'Alexander.

1. Introduction

1.1. Context. — In [6], Garoufalidis and Rozansky studied the rational vector space generated by the pairs (M, K) modulo orientation-preserving homeomorphism, where M is an *integral homology 3-sphere* ($\mathbb{Z}$HS), that is an oriented compact 3-manifold which has the same homology with integral coefficients as S^3, and K is an oriented knot in M. They defined a filtration on this space by means of null-moves, that are surgeries on claspers (see Garoufalidis, Goussarov and Polyak [4], and Habiro [8]) whose leaves are trivial in $H_1(M \setminus K; \mathbb{Z})$. They studied this filtration with the Kricker lift of the Kontsevich integral defined in [5]. The first step in the study of this filtration is the determination of the classes of pairs (M, K) up to null-moves. As a corollary of results of Matveev [11], Naik and Stanford [14], and Trotter [15], Garoufalidis and Rozansky established that two pairs (M, K) as above can be obtained from one another by a finite sequence of null-moves if and only if they admit S-equivalent Seifert matrices, and if and only if they have isomorphic integral Alexander modules and Blanchfield forms.

In this article, we consider pairs (M, K), where M is a *rational homology 3-sphere* ($\mathbb{Q}$HS), *i.e.,* an oriented compact 3-manifold which has the same homology with rational coefficients as S^3, and K is a *null-homologous knot* in M, *i.e.,* an oriented knot whose class in $H_1(M; \mathbb{Z})$ is trivial. We define the null Lagrangian-preserving surgeries, which play the role played by the null-moves in the integral case. We prove that the classes of pairs (M, K) modulo null Lagrangian-preserving surgeries are characterized by the classes of rational S-equivalence of their Seifert matrices, or by the isomorphism classes of their rational Alexander modules equipped with their Blanchfield forms. Furthermore, we prove that a fixed isomorphism between rational Alexander modules which preserves the Blanchfield form can be realized, up to multiplication by a power of t, by a finite sequence of null Lagrangian-preserving surgeries. Null Lagrangian-preserving surgeries define a filtration of the rational vector space generated by pairs (M, K) modulo orientation-preserving homeomorphism. This article is a first step in the study of this filtration, that is useful in the study of equivariant finite type knot invariants.

In [6], Garoufalidis and Rozansky characterized the classes of pairs (M, K), made of a $\mathbb{Z}$HS M and an oriented knot $K \subset M$, modulo null-moves, but they did not treat the question of the realization of a fixed isomorphism. In this article, we consider integral null Lagrangian-preserving surgeries, which generalize the null-moves, and define the same filtration of the vector space generated by all pairs (M, K) modulo orientation-preserving homeomorphism. We prove that a fixed isomorphism between integral Alexander modules which preserves the Blanchfield form can be realized, up to multiplication by a power of t, by a finite sequence of integral null Lagrangian-preserving surgeries. Garoufalidis and Rozansky used their work to determine the graded space associated with the above filtration in the case of a trivial Alexander module. In order to study the general case of a possibly non trivial Alexander module, the realization result is essential.

When it does not seem to cause confusion, we use the same notation for a curve and its homology class. All knots will be oriented.

1.2. Alexander module and Blanchfield form. — We first recall the definition of the Alexander module and of the Blanchfield form. Let (M, K) be a *$\mathbb{Q}$SK-pair*, that is a pair made of a rational homology sphere M and a null-homologous knot K in M. Let $T(K)$ be a tubular neighborhood of K. The *exterior* of K is $X = M \setminus Int(T(K))$. Consider the natural projection $\pi : \pi_1(X) \to \frac{H_1(X;\mathbb{Z})}{torsion} \cong \mathbb{Z}$, and the covering map $p : \tilde{X} \to X$ associated with its kernel. The covering $\tilde{X}$ is the *infinite cyclic covering* of X. The automorphism group of the covering, $Aut(\tilde{X})$, is isomorphic to $\mathbb{Z}$. It acts on $H_1(\tilde{X};\mathbb{Q})$. Denoting the action of a generator τ of $Aut(\tilde{X})$ as the multiplication by t, we get a structure of $\mathbb{Q}[t^{\pm 1}]$-module on $\mathcal{A}(K) = H_1(\tilde{X};\mathbb{Q})$. This $\mathbb{Q}[t^{\pm 1}]$-module is the *Alexander module* of K. It is a torsion $\mathbb{Q}[t^{\pm 1}]$-module.

Definition 1.1. — Let (M, K) and (M', K') be $\mathbb{Q}$SK-pairs. Let $\xi : \mathcal{A}(K) \to \mathcal{A}(K')$ be an isomorphism. The *τ-class* of ξ is the set of the isomorphisms $\xi \circ m_k$ for $k \in \mathbb{Z}$, where m_k is the multiplication by t^k.

Note that the τ-class of ξ is composed of all the isomorphisms that can be obtained from ξ by composition by isomorphisms of $\mathcal{A}(K)$ or $\mathcal{A}(K')$ induced by automorphisms of the underlying coverings.

If (M, K) is a *$\mathbb{Z}$SK-pair*, *i.e.,* if M is a $\mathbb{Z}$HS, define the *integral Alexander module* $\mathcal{A}_{\mathbb{Z}}(K)$ as the $\mathbb{Z}[t^{\pm 1}]$-module $H_1(\tilde{X};\mathbb{Z})$, similarly. It is a torsion $\mathbb{Z}[t^{\pm 1}]$-module, but we will see in Section 5 that it has no $\mathbb{Z}$-torsion. Hence it