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Elmar Grosse-Klönne

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Société Mathématique de France
Institut Henri Poincaré, 11, rue Pierre et Marie Curie
75231 Paris Cedex 05, France
Tél : (33) 01 44 27 67 99 • Fax : (33) 01 40 46 90 96
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LOCALLY ALGEBRAIC AUTOMORPHISMS OF THE $\mathrm{PGL}_2(F)$-TREE AND $\mathfrak{o}$-TORSION REPRESENTATIONS

BY ELMAR GROSSE-KLÖNNE

ABSTRACT. — For a local field F and an Artinian local coefficient ring Λ with the same positive residue characteristic p we define, for any $e \in \mathbb{N}$, a category $\mathfrak{C}^{(e)}(\Lambda)$ of $\mathrm{GL}_2(F)$-equivariant coefficient systems on the Bruhat-Tits tree X of $\mathrm{PGL}_2(F)$. There is an obvious functor from the category of $\mathrm{GL}_2(F)$-representations over Λ to $\mathfrak{C}^{(e)}(\Lambda)$. If $F = \mathbb{Q}_p$ then $\mathfrak{C}^{(1)}(\Lambda)$ is equivalent to the category of smooth $\mathrm{GL}_2(\mathbb{Q}_p)$-representations over Λ generated by their invariants under a pro-p-Iwahori subgroup. For general F and e we show that the subcategory of all objects in $\mathfrak{C}^{(e)}(\Lambda)$ with trivial central character is equivalent to a category of representations of a certain subgroup of $\mathrm{Aut}(X)$ consisting of "locally algebraic automorphisms of level e". For $e = 1$ there is a functor from this category to that of modules over the (usual) pro-p-Iwahori Hecke algebra; it is a bijection between irreducible objects.

Finally, we present a parallel of Colmez' functor $V \mapsto \mathbf{D}(V)$: to objects in $\mathfrak{C}^{(e)}(\Lambda)$ (for *any* F) we assign certain étale (φ, Γ)-modules over an Iwasawa algebra $\mathfrak{o}[[\widehat{N}_{0,1}^{(1)}]]$ which contains the (usually considered) Iwasawa algebra $\mathfrak{o}[[N_0]]$. This assignment preserves finite generation.

RÉSUMÉ (*Automorphismes localement algébriques de l'arbre de* $\mathrm{PGL}_2(F)$ *et représentations de* $\mathfrak{o}$*-torsion*)

Soient F un corps local et Λ un anneau artinien local de même caractéristique residuelle p. Pour $e \in \mathbb{N}$ on definit une catégorie $\mathfrak{C}^{(e)}(\Lambda)$ de systèmes à coefficients dans l'arbre de Bruhat-Tits de $\mathrm{PGL}_2(F)$, équivariant sous l'action de $\mathrm{GL}_2(F)$. Il y a un foncteur de la catégorie des représentations de $\mathrm{GL}_2(F)$ sur Λ vers $\mathfrak{C}^{(e)}(\Lambda)$. Si $F = \mathbb{Q}_p$, il induit une équivalence entre $\mathfrak{C}^{(1)}(\Lambda)$ et la catégorie des représentations

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ELMAR GROSSE-KLÖNNE, Humboldt-Universität zu Berlin, Institut für Mathematik, Rudower Chaussee 25, 12489 Berlin, Germany • *E-mail :* gkloenne@math.hu-berlin.de

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lisses de $\mathrm{GL}_2(F)$, engendrées par leurs vecteurs invariants sous un sous-groupe pro-p Iwahori. Pour chaque F et e, la sous-catégorie des objects dans $\mathfrak{C}^{(e)}(\Lambda)$ à caractère central trivial est équivalente à la catégorie des représentations d'un sous-groupe de $\mathrm{Aut}(X)$: le groupe des automorphismes « localement algébriques de niveau e » Pour $e = 1$ il y a un foncteur de cette catégorie vers celle des modules sur l'algèbre de pro-p Iwahori usuelle ; c'est une bijection entre objects irréductibles.

Finalement, on propose un foncteur de $\mathfrak{C}^{(e)}(\Lambda)$ vers la catégorie des (φ, Γ)-modules sur une algèbre d'Iwasawa $\mathfrak{o}[[\widehat{N}_{0,1}^{(1)}]]$ qui contient l'algèbre d'Iwasawa usuelle $\mathfrak{o}[[N_0]]$.

1. Introduction

Let F be a local field with residue characteristic $p > 0$ and uniformizer $p_F \in \mathcal{O}_F$ generating the maximal ideal $\mathfrak{p}_F$ of the ring of integers $\mathcal{O}_F$. Let G be the group of F-rational points of a reductive algebraic group over F. An important tool in the smooth representation theory of G on vector spaces over the complex numbers $\mathbb{C}$ is the localization technique which has been systematically developed by Schneider and Stuhler in their work [7]. Assigning to a smooth (admissible, finite length) G-representation on a $\mathbb{C}$-vector space V and a simplex τ in the Bruhat-Tits building of G the space of invariants of V under a suitable open subgroup of G fixing τ, one obtains a G-equivariant (homological) coefficient system $\mathcal{F}_V$ on X. If this assignment is carried out with appropriate care then V can be recovered from $\mathcal{F}_V$ as $V = H_0(X, \mathcal{F}_V)$, and in this way, the study of smooth (admissible, finite length) complex G-representations is transformed into the study of coefficient systems on X—in a sense these coefficient systems are 'smaller' objects, accessible by the representation theory of finite groups. These constructions work well also for smooth G-representations on vector spaces over fields of positive characteristic different from p.

On the other hand, if we ask for smooth G-representations on vector spaces V over a field k of characteristic p, then analogous assignments $V \mapsto \mathcal{F}_V$ are much weaker in general; typically, they do not allow to recover V. There seems to be basically only one example class of smooth (and possibly supercuspidal/supersingular) G-representations over k for which the classical (complex) theory carries over to wide extent: this is the case where $G = \mathrm{GL}_2(\mathbb{Q}_p)$ (or $G = \mathrm{SL}_2(\mathbb{Q}_p)$, or $G = \mathrm{PGL}_2(\mathbb{Q}_p)$) and where the smooth G-representations considered are generated by their invariants under a pro-p-Iwahori subgroup $U_\sigma^{(1)}$ of G. Namely, the category of these smooth G-representations is equivalent to a category of G-equivariant coefficient systems on the Bruhat-Tits tree X of $\mathrm{PGL}_2(\mathbb{Q}_p)$ satisfying a simple and natural axiomatic (i.e., the category $\mathfrak{C}^{(1)}(k)$ below). See [5] or Theorem 2.3 below for the precise statement.

The purpose of this paper is to discuss similar concepts for the groups $G = \mathrm{GL}_2(F)$ for general F. As in [7] we fix an index $e \geq 1$ (the 'level') and, for an edge η of the Bruhat-Tits tree X of $\mathrm{PGL}_2(F)$, we consider the open subgroup $U_\eta^{(e)}$ of G 'of level e' which fixes η. We fix an edge σ of X. For a ring Λ let $\mathfrak{C}^{(e)}(\Lambda)$ denote the category of G-equivariant homological coefficient systems of Λ-modules $\mathscr{F}$ on X such that for any vertex x and any edge η with $x \in \eta$ the transition map $\mathscr{F}(\eta) \to \mathscr{F}(x)$ is injective, its image is $\mathscr{F}(x)^{U_\eta^{(e)}}$ and generates $\mathscr{F}(x)$ as a representation of the stabilizer of x in G. There is an obvious functor $V \mapsto \mathscr{F}_V^{(e)}$ from the category of G-representation on Λ-modules to the category $\mathfrak{C}^{(e)}$; it satisfies $\mathscr{F}_V^{(e)}(\sigma) = V^{U_\sigma^{(e)}}$. If $\Lambda = \mathbb{C}$ then the category of smooth, admissible, finite length G-representations over $\mathbb{C}$ generated by $V^{U_\sigma^{(e)}}$ embeds into a full subcategory of $\mathfrak{C}^{(e)}(\mathbb{C})$ by means of this functor $V \mapsto \mathscr{F}_V^{(e)}$. Therefore it is natural to ask for the relevance of the category $\mathfrak{C}^{(e)}(\Lambda)$ for arbitrary Λ. Is it equivalent to a suitable category of G-representations ?

Let $\mathfrak{C}_0^{(e)}(\Lambda)$ denote the subcategory of all $\mathscr{F} \in \mathfrak{C}^{(e)}(\Lambda)$ on which the action of G factors through $\mathrm{PGL}_2(F)$. The basic observation of the present paper is that the $\mathrm{PGL}_2(F)$-action on any $\mathscr{F} \in \mathfrak{C}_0^{(e)}(\Lambda)$ and also on its homology $H_0(X, \mathscr{F})$ naturally extends to a much larger group $\widehat{G}^{(e)}$ containing $\mathrm{PGL}_2(F)$ and contained in the automorphism group $\mathrm{Aut}(X)$ of the tree X. Briefly, an element of $\mathrm{Aut}(X)$ belongs to $\widehat{G}^{(e)}$ if and only if for any edge η of X it acts on the ball of radius $e + \frac{1}{2}$ around η like an element of $\mathrm{PGL}_2(F)$. We call the elements of $\widehat{G}^{(e)}$ locally algebraic automorphisms of X of level e. (1)

The subgroups $U_\eta^{(e)}$ of G have as natural analogs certain pro-p-subgroups $\widehat{U}_\eta^{(e)}$ of $\widehat{G}^{(e)}$. Given a $\widehat{G}^{(e)}$-representation V we assign to it the coefficient system $\widehat{\mathscr{F}}_V^{(e)} \in \mathfrak{C}_0^{(e)}(\Lambda)$ with $\widehat{\mathscr{F}}_V^{(e)}(\eta) = V^{\widehat{U}_\eta^{(e)}}$ for edges η. Let $\widehat{\mathfrak{R}}_0^{(e)}(\Lambda)$ denote the category of $\widehat{G}^{(e)}$-representations V over Λ generated by $V^{\widehat{U}_\sigma^{(e)}}$, and smooth when regarded as representations of $\widehat{U}_\sigma^{(e)}$. (We find it convenient *not* to work with a topology on $\widehat{G}^{(e)}$; as a consequence, we do *not* have available the concept of a smooth $\widehat{G}^{(e)}$-representation. Instead, the subgroups $\widehat{U}_\sigma^{(e)}$ (and their open subgroups) mimick the role which open subgroups play in usual smooth representation theory.) Let now $\mathfrak{o}$ be a complete discrete valuation ring with residue field k, and assume that Λ is an Artinian local $\mathfrak{o}$-algebra with residue field k. Then our first main theorem is the following (Theorem 3.5):

(1) In particular we see that the smooth, admissible, finite length $\mathrm{PGL}_2(F)$-representations V over $\mathbb{C}$ generated by $V^{U_\sigma^{(e)}}$ automatically carry an action by the larger group $\widehat{G}^{(e)}$. For $F \neq \mathbb{Q}_p$ this fails if $\mathbb{C}$ is replaced by k (as follows e.g., from [3]), and from the point of view of the present paper, the failure of this principle is the reason for, or the manifestation of, the difference between the smooth $\mathrm{PGL}_2(F)$-representation theory over $\mathbb{C}$ and over k.