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Benjamin Schmidt

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Bulletin de la Société Mathématique de France
Société Mathématique de France
Institut Henri Poincaré, 11, rue Pierre et Marie Curie
75231 Paris Cedex 05, France
Tél : (33) 01 44 27 67 99 • Fax : (33) 01 40 46 90 96
revues@smf.ens.fr • <http://smf.emath.fr/>

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ON THE BIRATIONAL GEOMETRY OF SCHUBERT VARIETIES

BY BENJAMIN SCHMIDT

ABSTRACT. — We classify all \mathbb{Q} -factorializations of (co)minuscule Schubert varieties by using their Mori dream space structure. As a corollary we obtain a description of all IH-small resolutions of (co)minuscule Schubert varieties generalizing results of Perrin. We improve his results by including algebraically closed fields of positive characteristic and cominuscule Schubert varieties. Moreover, the use of \mathbb{Q} -factorializations and Mori dream spaces simplifies the arguments substantially.

RÉSUMÉ (Sur la géométrie birationnelle des variétés de Schubert)

Nous décrivons toutes les petites contractions de Mori des variétés de Schubert (co)minuscules en utilisant leur structure d'espace de rêve de Mori (*Mori dream space*). Nous en déduisons une description de toutes les résolutions IH-petites des variétés de Schubert (co)minuscules et généralisons ainsi les résultats de Perrin : nous étendons ses résultats à tout corps algébriquement clos de caractéristique quelconque et aux variétés de Schubert cominuscules. En outre, l'utilisation des petites contractions des espaces de rêve de Mori simplifie grandement les arguments.

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BENJAMIN SCHMIDT, Department of Mathematics, The Ohio State University, 231 W 18th Avenue, Columbus, OH 43210-1174, USA • E-mail : schmidt.707@osu.edu

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1. Introduction

A fundamental goal of algebraic geometry is to describe birational models with better properties than the average variety. These models should be smooth or at least have mild singularities. A major step towards this goal was the resolution of singularities for any field of characteristic 0 by Hironaka in [5]. For Schubert varieties there are the well known Bott-Samelson resolutions introduced in [1]. They are rational resolutions and lead to a character formula for representations of reductive groups. On the other hand, Zelevinsky constructed IH-small resolutions (see Definition 5.1) in the case of Schubert varieties in Grassmannians and used them to compute Kazhdan-Lusztig polynomials (see [15]). Sankaran and Vanchinathan obtained similar results in Lagrangian and maximal isotropic Grassmannians in [10] and [11].

Many classical results on Schubert varieties in Grassmannians can be generalized to minuscule or cominuscule Schubert varieties (see Definition 3.2). In [8] Perrin gives a complete classification of all IH-small resolutions of minuscule Schubert varieties over \mathbb{C} . This was done using a connection to the minimal model program: Totaro proved that any IH-small resolution is a relative minimal model in [14, Proposition 8.3]. Perrin was able to classify all relative minimal models of minuscule Schubert varieties. Since most of the results from the minimal model program are only known in characteristic 0, this approach is only valid over the complex numbers.

We will investigate further into the birational geometry of Schubert varieties over an arbitrary algebraically closed field using a different approach. A Mori-small morphism from a normal and \mathbb{Q} -factorial variety to a normal variety is called a \mathbb{Q} -factorialization. Our goal is to determine all \mathbb{Q} -factorializations of any (co)minuscule Schubert varieties.

In order to handle the occurring combinatorics in the Weyl group one defines a quiver for each reduced expression (see Definition 3.1). Due to a result in [13], every reduced expression of a (co)minuscule element is unique up to commuting relations. This implies that there is a unique quiver associated to each (co)minuscule Schubert variety. Moreover, there is an explicit combinatorial description of the quivers of minuscule elements (see Theorem 3.3). This provides a very concrete object to work with. We define a partial ordering on the vertices of the quiver, call the maximal elements peaks and assign each vertex a value called the height.

For each ordering of the peaks, there is a birational projective morphism $\widehat{\pi} : \widehat{X}(\widehat{w}) \rightarrow X(w)$ (see Section 3). The varieties $\widehat{X}(\widehat{w})$ are towers of locally

trivial fibrations with fibers being Schubert varieties. This generalizes the Bott-Samelson resolution which is a tower of locally trivial \mathbb{P}^1 -fibrations. These varieties $\widehat{X}(\widehat{w})$ are generally not smooth, but in our case always locally \mathbb{Q} -factorial and normal. We prove the following theorem.

THEOREM 4.2. — *Let $X(w)$ be a (co)minuscule Schubert variety. Then all \mathbb{Q} -factorializations of $X(w)$ are given by the morphisms $\widehat{\pi} : \widehat{X}(\widehat{w}) \rightarrow X(w)$ obtained from any ordering of the peaks.*

The use of Mori dream spaces is the main ingredient for proving this result. These spaces are tailor-made for running the minimal model program in any characteristic (see [6]). First, we show that the varieties $\widehat{X}(\widehat{w})$ are indeed Mori dream spaces. More precisely, taking all $\widehat{X}(\widehat{w})$ for any possible ordering of the peaks leads to all the small \mathbb{Q} -factorial modifications defining a Mori dream space. Following [4] and [8], we give explicit descriptions of the nef and effective cones of divisors using the structure of towers of locally trivial fibrations. This is all that is needed to describe the Mori dream space structure. The Theorem follows because the morphisms $\widehat{\pi} : \widehat{X}(\widehat{w}) \rightarrow X(w)$ are all Mori-small, i.e., they do not contract any divisor.

Since any IH-small morphism is also Mori-small, classifying all IH-small resolutions of (co)minuscule Schubert varieties over any algebraically closed field becomes a matter of checking which of these \mathbb{Q} -factorializations are IH-small. Zelevinsky for Grassmannians and Perrin for minuscule homogeneous spaces define specific orderings using heights of peaks that are called neat. Generalizing their results by including cominuscule Schubert varieties and algebraically closed fields of positive characteristic, we obtain the following corollary.

COROLLARY 5.6. — *Let $X(w)$ be a (co)minuscule Schubert variety over an algebraically closed field k . Then the IH-small resolutions of $X(w)$ are exactly given by the morphisms $\widehat{\pi} : \widehat{X}(\widehat{w}) \rightarrow X(w)$, where \widehat{w} is obtained from a neat ordering of the peaks and $\widehat{X}(\widehat{w})$ is smooth.*

Note that there is an explicit combinatorial criterion for smoothness of the varieties $\widehat{X}(\widehat{w})$ (see Section 5). Using Theorem 4.2 allows for a uniform treatment of both the minuscule and cominuscule case. While similar arguments as in [8] can be used at least over \mathbb{C} , the geometry is slightly different. This results in even more complicated combinatorics (see [12]). Therefore, we strongly believe that the present proof is much more suitable to approach non (co)minuscule cases.

Section 2 sets up some basic notation. In Section 3 we recall (co)minuscule Schubert varieties and the definition of $\widehat{X}(\widehat{w})$. Section 4 is concerned with the proof of Theorem 4.2, while Section 5 presents Corollary 5.6.