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SMOOTHENING CONE POINTS WITH RICCI FLOW

BY DANIEL RAMOS

ABSTRACT. — We consider Ricci flow on a closed surface with cone points. The main result is: given a (nonsmooth) cone metric g_0 over a closed surface there is a smooth Ricci flow $g(t)$ defined for $(0,T]$, with curvature unbounded above, such that $g(t)$ tends to g_0 as $t \to 0$. This result means that Ricci flow provides a way for instantaneously smoothening cone points. We follow the argument of P. Topping in [11] modifying his reasoning for cusps of negative curvature; in that sense we can consider cusps as a limiting zero-angle cone, and we generalize to any angle between 0 and 2π.

RÉSUMÉ (*Lissage de points coniques avec flot de Ricci*). — On considère un flot de Ricci sur une surface fermée avec des points coniques. Le résultat principal est : étant donné une métrique conique g_0 (non lisse) sur une surface fermée, il existe un flot de Ricci lisse $g(t)$ défini pour $(0,T]$, avec courbure non bornée supérieurement, tel que $g(t)$ tend vers g_0 quand $t \to 0$. Cet résultat implique que le flot de Ricci donne une méthode pour lisser instantanément des points coniques. On suit un argument de P. Topping dans [11] en modifiant son raisonnement pour les cusps de courbure négative; en ce sens, on peut considérer les cusps comme un cas limite de points coniques d'angle zéro, et nous généralisons à un angle quelconque entre 0 et 2π.

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1. Introduction

Ricci flow on closed surfaces was studied first by R. Hamilton [7] and B. Chow [3]. Hamilton proved that any smooth closed Riemannian surface $(\mathcal{M}, g_0)$ admits a volume-normalized Ricci flow $g(t)$, $t \in [0, T]$, with uniformly bounded curvature, having g_0 as initial condition, $g(0) = g_0$. Hamilton and Chow proved that this flow is unique and well defined and converges to a constant curvature metric for any initial condition g_0. This gives a proof for the uniformization theorem, [1] (see also [4]). The unnormalized flow may develop finite-time singularities in the case of a sphere, when the curvature tends globally to infinity as well as the diameter tends to zero. Some analogous results were obtained for Ricci flow on orbifold surfaces by L-F. Wu and Chow [12], [5], [2]. These authors assume an equivariant definition of the Ricci flow under the action of the isotropy group of the cone points. Therefore, the only nontrivial cases are bad orbifolds, which do not admit a smooth manifold as global branched covering space, so the Ricci flow cannot be lifted there. They prove that bad orbifolds (the teardrop and the football) admit a (normalized) Ricci flow converging to a soliton solution.

The alternative consideration of the Ricci flow just acting on the smooth part of the orbifold leads to consider the Ricci flow on an open, noncomplete manifold, which does not fit in the classical theory of Hamilton, so existence and uniqueness might be lost. H. Yin obtained, however, an existence theorem for Ricci flow on cone surfaces [13], with uniformly bounded curvature, defined on the smooth part of the surface, and preserving the conical structure of each singular point. This is the analogous to the classical and orbifold Ricci flow. On a different approach, Topping and G. Giesen [6] obtained an existence theorem for Ricci flow on incomplete surfaces, which becomes instantaneously complete but might have unbounded curvature from below. This exposes the nonuniqueness of solutions.

In another work, Topping [11] considered a complete open surface with cusps of negative curvature and proved the existence of a instantaneously smooth Ricci flow with unbounded curvature, a "smoothening flow" which erases instantaneously the cusps. This requires a generalized notion of initial metric for a flow, that we will use thorough the paper:

Definition 1 (Cf. [11] Definition 1.1). — *Let $\mathcal{M}$ be a smooth manifold, and $p_1, \ldots, p_n \in \mathcal{M}$. Let g_0 be a Riemannian metric on $\mathcal{M} \setminus \{p_1, \ldots, p_n\}$ and let $g(t)$ be a smooth Ricci flow on $\mathcal{M}$ for $t \in (0, T]$. We say that $g(t)$ has initial condition g_0 if*

$$g(t) \longrightarrow g_0 \text{ as } t \to 0$$

smoothly locally on $\mathcal{M} \setminus \{p_1, \ldots, p_n\}$.

The technique for this result consists in capping the cusps of the original metric g_0 with a smooth part near the cusp point, in an increasing sequence of metrics, each term with a further and smaller capping. This sequence of smooth metrics gives rise to a sequence of (classical) Ricci flows, and the work consists in proving that this sequence has a limiting Ricci flow on $\mathcal{M}$ which has g_0 as initial condition in the sense of Definition 1. Our work proves that this technique works equally well on cone surfaces, using truncated or "blunt" cones as approximations for a cone point. In our setting, cusps would be seen as a limiting case of a zero-angle cone. This provides an instantaneously smooth Ricci flow that smooths out the cone points of a cone surface.

This technique of approximating a nonsmooth manifold by smooth ones and limiting the sequence of corresponding Ricci flows has also been used by M. Simon [10] in dimension three to investigate the Gromov-Hausdorff limit of sequences of three-manifolds. Another similar development has been recently done by T. Richard [9] using this technique for smoothening out a broader class of Alexandrov surfaces.

The paper is organized as follows: in Sections 2 and 3 we review the notions of cone surface and Ricci flow on cone surfaces, and state the two main theorems of the paper (existence and uniqueness of the smoothening flow). In Section 4 we build the truncated cones that will serve us as approximations of a cone point and in Section 5 we build upper barriers that, applied to our truncated cones, will give us control on the convergence of the sequence. In Section 6 we put together the preceding lemmas to prove the existence theorem; and finally in Section 7 we prove the uniqueness theorem.

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2. Cone points

Cone surfaces are topological surfaces equipped with a Riemannian metric which is smooth everywhere except on some discrete set of points (*cone points*) that look like the vertex of a cone. Typical examples include orbifolds, where a group acting by isometries leads to identification of different directions as seen from a fixed point. In the case of two dimensions, orientable orbifolds consist locally in the quotient of a smooth manifold by perhaps the action of a cyclic group acting by rotations, leading to the rise of singular points at the center of the rotations. The space of directions is no longer a metric circle of length 2π but a metric circle of length $\frac{2\pi}{n}$ (this is the cone angle). General 2-dimensional cone points include all angles, not only submultiples of 2π, although we will