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RIGIDITY OF REDUCIBILITY OF GEVREY QUASI-PERIODIC COCYCLES ON $U(n)$

BY XUANJI HOU & GEORGI POPOV

ABSTRACT. — We consider the reducibility problem of cocycles (α, A) on $\mathbb{T}^d \times U(n)$ in Gevrey classes, where α is a Diophantine vector. We prove that, if a Gevrey cocycle is conjugated to a constant cocycle (α, C) by a suitable measurable conjugacy $(0, B)$, then for almost all C it can be conjugated to (α, C) in the same Gevrey class, provided that A is sufficiently close to a constant. If B is continuous we obtain that it is Gevrey smooth. We consider as well the global problem of reducibility in Gevrey classes when $d = 1$.

RÉSUMÉ (*Rigidité de réductibilité des cocycles quasi-périodiques de Gevrey sur $U(n)$*)
On considère le problème de la réductibilité de cocycles (α, A) sur $\mathbb{T}^d \times U(n)$ dans les classes de Gevrey, où α est Diophantien. Si A est proche d'une constante et le Gevrey cocycle (α, A) est conjuqué au cocycle constant (α, C) par une conjugaison mesurable $(0, B)$, on montre que pour presque tous C le cocycle peut êtrte conjuguer à (α, C) dans la même classe de Gevrey . Si B est continue on obtient qu'elle est Gevrey. On considère aussi le problème de la réductibilité globale dans les classes de Gevrey dans le cas où $d = 1$.

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1. Introduction

This article is concerned with the reducibility of cocycles in Gevrey classes on the unitary group $U(n)$. A cocycle on $U(n)$ is a diffeomorphism of $\mathbb{T}^d \times U(n)$, $\mathbb{T}^d$ being the torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$, given by the skew-product

$$\begin{aligned}(\alpha, A):\ & \mathbb{T}^d \times \mathbb{C}^n \to \mathbb{T}^d \times \mathbb{C}^n \\ & (\theta, v) \mapsto (\theta + \alpha, A(\theta)v),\end{aligned}$$

where $\alpha \in \mathbb{T}^d$ and $A : \mathbb{T}^d \to U(n)$ is a map. The corresponding dynamics is defined by the iterates of the cocycle by composition $(\alpha, A)^n$, $n \in \mathbb{Z}$. We denote by $C^r(\mathbb{T}^d, U(n))$ $(r = 0, 1, \dots, \infty, \omega)$ the set of all C^r functions A. For any $\rho \geq 1$ and $L > 0$ we denote by $\mathscr{G}^\rho_L(\mathbb{T}^d, U(n))$ the class of Gevrey$-\mathscr{G}^\rho$ functions with an exponent ρ and Gevrey constant L. A map $A \in C^\infty(\mathbb{T}^d, U(n))$ belongs to that class if it satisfies (2.10) (see Section 2.2). Denote by $SW^{\mathscr{G}}_\rho(\mathbb{T}^d, U(n))$ $(SW^r(\mathbb{T}^d, U(n)))$, the set of all Gevrey$-\mathscr{G}^\rho$ (C^r) quasi-periodic cocycles on $U(n)$.

The dynamics is particularly simple if (α, A) is a constant cocycle. The cocycle (α, A) is said to be constant if A is a constant matrix. Two cocycles $(\alpha, A), (\alpha, \widetilde{A}) \in SW^r(\mathbb{T}^d, U(n))$ are said to be conjugated if there exists $B : \mathbb{T}^d \to U(n)$ such that

$$Ad(B).(\alpha, A) := (\alpha, B(\cdot + \alpha)^{-1} A B) = (\alpha, \widetilde{A}),$$

which means that $B(\theta + \alpha)^{-1} A(\theta) B(\theta) = \widetilde{A}(\theta)$ for any $\theta \in \mathbb{T}^d$. The cocycle (α, A) is said to be reducible if it is conjugated to a constant one. We say also that the conjugation or the reducibility is Gevrey$-\mathscr{G}^\rho$, C^r, or measurable, if B belongs to the corresponding class of functions.

Reducibility problem of cocycles has been investigated for a long time. The local reducibility problem (the cocycle is close to a constant one) is usually studied using KAM-type iterations. In particular, Eliasson's KAM method developed in [3] gives full-measure reducibility for generic one-parameter families of cocycles [2, 4, 10, 9, 5, 6]. The global reducibility problem (cocycles are no longer close to a constant one) has been studied by Avila, Krikorian and others. By means of a renormalization scheme Krikorian obtained a global density result for C^∞ cocycles on $SU(2)$ [11] and also results for cocycles on $SL(2, \mathbb{R})$ [1, 12]. Almost reducibility for Gevrey cocycles has been studied by Chavaudret in [2].

The *rigidity problem* we are interested in, can be formulated as follows. Suppose that a Gevrey$-\mathscr{G}^\rho$ cocycle is measurably reducible. Is it also Gevrey$-\mathscr{G}^\rho$ reducible? In the case of C^∞ or C^ω cocycles the rigidity problem has been investigated in [1, 12, 7, 6].

In this paper, we will focus our attention on the Gevrey case. We will prove a local rigidity result of reducibility in Gevrey classes which can be viewed as a Gevrey analogue of the main result in [7]. To this end we use techniques developed in [17]. When $d = 1$, the local result together with Krikorian's renormalization scheme imply as in [11, 1] a global rigidity result for Gevrey quasi-periodic cocycles on $\mathbb{T}^1 \times U(n)$.

Why are we interested in Gevrey classes? Gevrey classes appear naturally in the KAM theory when dealing with Diophantine frequencies [16, 17]. They provide a natural framework for studying KAM systems, Birkhoff normal forms with an exponentially small reminder terms and the Nekhoroshev theory, and give an inside relation between these theories [14, 15, 16, 17]. One can consider as well the more general Roumieu classes of non-quasi-analytic functions. In the case of Bruno-Rüssmann arithmetic conditions we suggest that similar results hold in appropriate Roumieu spaces.

To formulate the main results we recall certain arithmetic conditions. Given $\gamma > 0$ and $\tau > d - 1$, we say that $\alpha \in \mathbb{R}^d$ is (γ, τ)-Diophantine if

$$|e^{2\pi i \langle k, \alpha \rangle} - 1| > \frac{\gamma^{-1}}{|k|^\tau}, \quad 0 \neq k \in \mathbb{Z}^d, \tag{1.1}$$

and we denote by $\mathrm{DC}\,(\gamma, \tau)$ the set of all such Diophantine vectors. Hereafter, $i := \sqrt{-1}$ stands for the imaginary unit. It is well known that $\mathrm{DC}\,(\tau) := \bigcup_{\gamma > 0} \mathrm{DC}\,(\gamma, \tau)$ is a set of full Lebesgue measure. For any given $\alpha \in \mathbb{R}^d$, we denote by $\Upsilon(\alpha; \chi, \nu)$ the set of all vectors $(\phi_1, \dots, \phi_n) \in \mathbb{R}^n$, satisfying

$$|\langle k, \alpha \rangle + \phi_p - \phi_q - j| \geq \frac{\chi}{(1 + |k|)^\nu} \tag{1.2}$$

for any $p \neq q \in \{1, 2, \dots, n\}$, $k \in \mathbb{Z}^d$ and $j \in \mathbb{Z}$. The set

$$\Upsilon(\alpha) := \bigcup_{\chi, \nu > 0} \Upsilon(\alpha; \chi, \nu)$$

has full Lebesgue measure in $\mathbb{R}^n$. Recall that the Lie group $U(n)$ consists of all $A \in GL(n, \mathbb{C})$ satisfying $A^* A = I$. Hereafter, I stands for the identity matrix and A^* is the adjoint matrix to A in $M_n = M_n(\mathbb{C})$. The corresponding Lie algebra $u(n)$ is the set of $X \in gl(n, \mathbb{C})$ satisfying $X^* + X = 0$. Any $A \in U(n)$ is diagonalizable, and the set of eigenvalues of A, denoted by $\mathrm{Spec}\,(A)$, is a subset of $\{z \in \mathbb{C} : |z| = 1\}$. Denote by $\Sigma(\alpha; \chi, \nu)$ the set of $A \in U(n)$ with spectrum $\mathrm{Spec}\,(A) := \{\lambda_1, \lambda_2, \dots, \lambda_n\}$ satisfying

$$|\lambda_p - \lambda_q e^{2\pi i \langle k, \alpha \rangle}| \geq \frac{\chi}{(1 + |k|)^\nu} \tag{1.3}$$