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## **EXAMPLES OF RATIONAL MAPS OF $\mathbb{CP}^2$ WITH EQUAL DYNAMICAL DEGREES AND NO INVARIANT FOLIATION**

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## EXAMPLES OF RATIONAL MAPS OF $\mathbb{CP}^2$ WITH EQUAL DYNAMICAL DEGREES AND NO INVARIANT FOLIATION

BY SCOTT R. KASCHNER, RODRIGO A. PÉREZ  
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ABSTRACT. — We present simple examples of rational maps of the complex projective plane with equal first and second dynamical degrees and no invariant foliation.

RÉSUMÉ (*Exemple d'applications rationnelles de  $\mathbb{CP}^2$  à degrés dynamiques égaux sans feuilletage invariant*)

Nous présentons des exemples simples de application rationnelle du plan projectif complexe avec l'égalité des premier et second degrés dynamiques et pas de feuilletage invariant.

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## 1. Introduction

A meromorphic map  $\varphi : X \dashrightarrow X$  of a compact Kähler manifold  $X$  induces a well-defined pullback action  $\varphi^* : H^{p,p}(X) \rightarrow H^{p,p}(X)$  for each  $1 \leq p \leq \dim(X)$ . The  $p$ -th dynamical degree

$$(1) \quad \lambda_p(\varphi) := \lim_{n \rightarrow \infty} \|(\varphi^n)^* : H^{p,p}(X) \rightarrow H^{p,p}(X)\|^{1/n}$$

describes the asymptotic growth rate of the action of iterates of  $\varphi$  on  $H^{p,p}(X)$ . Originally, the dynamical degrees were introduced by Friedland [22] and later by Russakovskii and Shiffman [34] and shown to be invariant under birational conjugacy by Dinh and Sibony [19]. Note that dynamical degrees were originally defined with the limit in (1) replaced by limsup. However, it was shown in [19, 18] that the limit always exists.

One says that  $\varphi$  is *cohomologically hyperbolic* if one of the dynamical degrees is strictly larger than all of the others. In this case, there is a conjecture [25, 27] that describes the ergodic properties of  $\varphi$ . This conjecture has been proved in several particular sub-cases [26, 13, 17].

What happens in the non-cohomologically hyperbolic case? If a meromorphic map preserves a fibration, then there are nice formulae relating the dynamical degrees of the map [15, 16]. This is the case in the following two examples:

- a) It was shown in [14] that bimeromorphic maps of surfaces that are not cohomologically hyperbolic ( $\lambda_1(\varphi) = \lambda_2(\varphi) = 1$ ) always preserve an invariant fibration.
- b) Meromorphic maps that are not cohomologically hyperbolic arise naturally when studying the spectral theory of operators on self-similar spaces [35, 3, 24, 2]. All the examples studied in that context preserve an invariant fibration. In several cases, this fibration made it significantly easier to compute the limiting spectrum of the action.

Based on this evidence, Guedj asked in [27, p. 103] whether every non-cohomologically hyperbolic map preserves a fibration.

We will prove:

**THEOREM 1.1.** — *The rational map  $\varphi : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$  given by*

$$(2) \quad \varphi[X : Y : Z] = [-Y^2 : X(X - Z) : -(X + Z)(X - Z)]$$

*is not cohomologically hyperbolic ( $\lambda_1(\varphi) = \lambda_2(\varphi) = 2$ ), and no iterate of  $\varphi$  preserves a singular holomorphic foliation. Moreover, for a Baire generic set of automorphisms  $A \in \mathrm{PGL}(3, \mathbb{C})$ , the composition  $A \circ \varphi^4$  has the same properties.*

Since preservation of a fibration is a stronger condition than preservation of a singular foliation,  $\varphi$  provides an answer to the question posed by Guedj.

After reading a preliminary version of this paper, Charles Favre asked if the same behavior can be found for a polynomial map of  $\mathbb{C}^2$ . In [20, §7.2], there is a list of seven types of non-cohomologically hyperbolic polynomial mappings. Our method of proof does not apply in most of these examples for trivial reasons, but we found a map of type (3) for which the same result holds:

**THEOREM 1.2.** — *The polynomial map*

$$(3) \quad \psi(x, y) := (x(x - y) + 2, (x + y)(x - y) + 1)$$

*extends as a rational map  $\psi : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$  that is not cohomologically hyperbolic ( $\lambda_1(\psi) = \lambda_2(\psi) = 2$ ), and no iterate of  $\psi$  preserves a singular holomorphic foliation on  $\mathbb{CP}^2$ .*

Non-cohomologically hyperbolic maps of 3-dimensional manifolds  $X$  arise naturally as certain pseudo-automorphisms that are “reversible” on  $H^{1,1}(X)$  [5, 6, 32]. For these mappings it follows from Poincaré duality that  $\lambda_1(\varphi) = \lambda_2(\varphi)$ . Recently, Bedford, Cantat, and Kim [4] have found a reversible pseudo-automorphism of an iterated blow-up of  $\mathbb{CP}^3$  which does not preserve any invariant foliation. It also answers the question posed by Guedj.

Many authors have studied meromorphic (and rational) maps that preserve foliations and algebraic webs, including [21, 8, 31, 11, 12, 7, 14]. We will provide a direct proof of Theorem 1.1 rather than appealing to results from previous papers. It has the advantage of being self-contained, and of providing insight on the *mechanism* that prevents  $\varphi$  from preserving a foliation.

Let us give a brief idea of this mechanism. The fourth iterate  $\varphi^4$  has an indeterminate point  $p$  that is blown-up by  $\varphi^4$  to a singular curve  $C$ . Any foliation  $\mathcal{F}$  must be either generically transverse to  $C$  or have  $C$  as a leaf. This allows us to show that  $(\varphi^4)^*\mathcal{F}$  must be singular at either at  $p$  or at some preimage  $r$  of the singular point. Both of these points have infinite  $\varphi^4$  pre-orbits, at each point of which  $\varphi^{4n}$  is a finite map. This generates a sequence of distinct points  $\{a_{-n}\}_{n=0}^\infty$  such that  $(\varphi^{4n})^*\mathcal{F}$  is singular at  $a_{-n}$ . If  $(\varphi^\ell)^*\mathcal{F} = \mathcal{F}$  for some  $\ell$  this implies that  $\mathcal{F}$  is singular at infinitely many points, providing a contradiction. The same method of proof applies for the polynomial map in Theorem 1.2.

**QUESTION 1.3.** — *Our proof of Theorem 1.2 relies on the compactness of  $\mathbb{CP}^2$ . Does  $\psi$  preserve a foliation on  $\mathbb{CP}^2$ ?*