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A NEW TWO-PARAMETER FAMILY OF ISOMONODROMIC DEFORMATIONS OVER THE FIVE PUNCTURED SPHERE

by Arnaud Girand

Abstract. — The object of this paper is to describe an explicit two-parameter family of logarithmic flat connections over the complex projective plane. These connections have dihedral monodromy and their polar locus is a prescribed quintic composed of a conic and three tangent lines. By restricting them to generic lines we get an algebraic family of isomonodromic deformations of the five-punctured sphere. This yields new algebraic solutions of a Garnier system. Finally, we use the associated Riccati one-forms to construct and prove the integrability (in the transversally projective sense) of a subfamily of Lotka-Volterra foliations.

Résumé (*Une nouvelle famille à deux paramètres de déformations isomonodromiques sur la sphère à cinq trous*)

Le but de cet article est de décrire une famille explicite à deux paramètres de connexions logarithmiques plates au dessus du plan projectif complexe. Ces connexions sont à monodromie diédrale et leur lieu polaire est une quintique prescrite, composée d'une conique et de trois droites tangentes. Par restriction aux droites génériques, on obtient alors une famille algébrique de déformations isomonodromiques de la sphère à cinq trous. Ceci livre de nouvelles solutions algébriques d'un système de Garnier. Enfin, nous utilisons les formes de Riccati associées à ces connexions pour construire et montrer l'intégrabilité (au sens transversalement projectif) d'une sous-famille de feuilletages de Lotka-Volterra.

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1. Introduction

In this section we describe the main result of this paper, namely an explicit construction of a two-parameter family of logarithmic flat connections over the complement of a particular quintic curve in $\mathbb{P}^2_{\mathbb{C}}$. The restriction of any element of this family to generic lines in the projective plane gives an isomonodromic deformation over the five punctured sphere, to which we can associate an algebraic solution of some Hamiltonian system of partial differential equations, namely the Garnier-2 system.

1.1. Topology of the complement of a particular plane quintic. — In this paper, we concern ourselves with setting up a two-parameter family of logarithmic flat $\mathfrak{sl}_2(\mathbb{C})$-connections over $\mathbb{P}^2$ with a specific polar locus, namely a quintic curve $\mathcal{Q}$ composed of a circle and three tangent lines. More precisely, in homogeneous coordinates $[x : y : t]$, $\mathcal{Q}$ is defined, up to $PGL_3(\mathbb{C})$ action, by the equation

$$xyt(x^2 + y^2 + t^2 - 2(xy + xt + yt)) = 0 \ .$$

Before stating our main result, let us specify what we are looking for: we want to find a family of rank two logarithmic flat connections over $\mathbb{P}^2$ with polar locus equal to some small degree curve and "interesting monodromy". We will show that it is possible to do so with the quintic $\mathcal{Q}$ define above.

Definition 1.1. — We say that the monodromy representation associated with a rank two logarithmic flat $\mathfrak{sl}_2(\mathbb{C})$-connection over $\mathbb{P}^2 - \mathcal{Q}$ is non-degenerate if

- its image forms an irreducible subgroup of $SL_2(\mathbb{C})$;
- its local monodromy (see Definition 2.3) around any irreducible component of $\mathcal{Q}$ is projectively non-trivial (i.e is non-trivial in $PSL_2(\mathbb{C})$).

In order to establish the existence of such representations, we use the following result by Degtyarev.

Proposition 1.2 (Degtyarev, 1999 [5]). — *The fundamental group* Γ *of the complement of a smooth conic and three tangent lines in* $\mathbb{P}^2$ *admits the following presentation:*

$$\Gamma \cong \langle a, b, c \,|\, (ab)^2(ba)^{-2} = (ac)^2(ca)^{-2} = [b, c] = 1 \rangle \ .$$

More precisely, Degtyarev proves that we can take a (resp. b, c) to be a loop realising the local monodromy (see Definition 2.3) around the conic $\mathscr{C} := (x^2 + y^2 + t^2 - 2(xy + xt + yt) = 0)$ (resp. the lines $(y = 0)$, $(x = 0)$), as illustrated in the left-hand side of Fig. 1. Also note that the fundamental group of the intersection of $\mathbb{P}^2 - \mathcal{Q}$ with any generic line is isomorphic to the free group $\mathbf{F}_4 := \langle d_1, \ldots, d_5 \,|\, d_1 \ldots d_5 = 1 \rangle$; the Lefschetz hyperplane theorem (see [15],

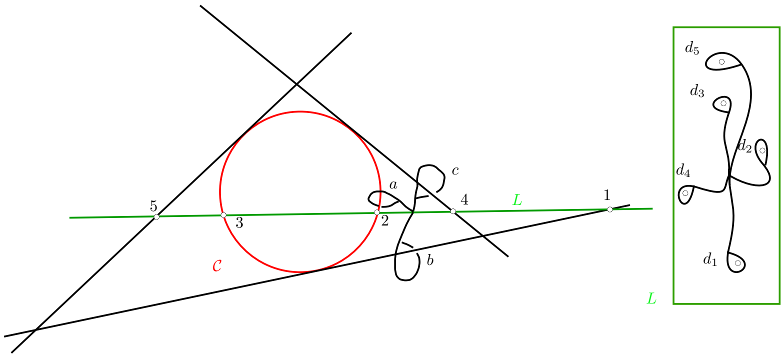

FIGURE 1. Fundamental group of the complement of the quintic Q in $\mathbb{P}^2$ and restriction to a generic line.

Theorem 7.4) tells us that the natural morphism $\tau : \mathbf{F}_4 \to \Gamma$ is onto. Moreover we know from the explicit Van Kampen method given in Subsection 4.1 of [5] that the group Γ can be computed by taking the four free generators of the fundamental group of the intersection of $\mathbb{P}^2 - Q$ with any generic line and adding some braid monodromy relations. Thus, if we chose a line going through the base point used to define a, b and c then τ is given (up to a permutation of the d_i) by (see the right-hand side of Fig. 1):

$$\begin{aligned} d_1 &\mapsto b \\ d_2 &\mapsto a \\ d_3 &\mapsto bab^{-1} \\ d_4 &\mapsto c \\ d_5 &\mapsto (abac)^{-1} \ . \end{aligned}$$

In particular, any non-degenerate representation ρ of Γ must satisfy

$$\rho(a), \rho(b), \rho(c), \rho(abac) \neq \pm I_2 \ .$$

PROPOSITION 1.3. — *The only (up to conjugacy) family of non-degenerate representations of* Γ *into* $SL_2(\mathbb{C})$ *is as follows:*

$$\rho_{u,v} : a \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad b \mapsto \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}, \quad c \mapsto \begin{pmatrix} v & 0 \\ 0 & v^{-1} \end{pmatrix}, \ \textit{for } u, v \in \mathbb{C}^* \ .$$

1.2. Main results. — The core of this paper will be devoted to proving the following theorem, in which we explicitly construct the announced family of rank two logarithmic flat connections.