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MULTI-MICROLOCALIZATION AND MICROSUPPORT

by Naofumi Honda, Luca Prelli & Susumu Yamazaki

Abstract. — The purpose of this paper is to establish the foundations of multi-microlocalization, in particular, to give the fiber formula for the multi-microlocalization functor and estimate of microsupport of a multi-microlocalized object. We also give some applications of these results.

Résumé (*Multi-microlocalisation et microsupport*). — Le but de cet article est d'établir les fondements de la multi-microlocalisation, en particulier, de donner une formule de fibre pour le foncteur de multi-microlocalisation et aussi une estimation du microsupport d'un objet multi-microlocalisé. Nous donnons également quelques applications de ces résultats.

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Introduction

A microlocalized object of a sheaf F along a closed submanifold M was first introduced in M. Sato, T. Kawai and M. Kashiwara [9], which is locally described by, roughly speaking, local cohomology groups of F with support in a dual cone of the edge M (see also M. Kashiwara and P. Schapira [4]). As a result it can be tightly related with, via Čech cohomology groups, boundary values of local sections of F defined on open cones of the edge M. It is well known that, for example, Sato's microfunctions, which are obtained by applying the microlocalization functor along a real analytic manifold M to the sheaf of holomorphic functions, can be regarded as boundary values of holomorphic functions locally defined on wedges with the edge M.

We sometimes, in study of partial differential equations, need to consider a boundary value of a function defined on a cone along a family χ of several closed submanifolds. In such a study, J. M. Delort [1] had introduced *simultaneous microlocalization* along a normal crossing divisor χ which gives a boundary value of a function defined on a dual poly-sector. *Bi-microlocalization* along submanifolds $\chi = \{M_1, M_2\}$ with $M_1 \subset M_2$ was also introduced by P. Schapira and K. Takeuchi [10] and [11] which defines a different kind of a boundary value.

On the other hand, in the paper [3], the first and the second authors of this article established the notion of *the multi-normal cone* for a family χ of closed submanifolds with a suitable configuration, and they also constructed *the multi-specialization functor* along χ. One can observe that cones appearing in simultaneous microlocalization and bi-microlocalization are characterized by using the multi-normal cone and that both microlocalization functors coincide with the *multi-microlocalization functor* along χ where the latter functor is obtained by repeated application of Sato's Fourier transformation to the multi-specialization functor. Hence multi-microlocalization gives us a uniform machinery for both simultaneous microlocalization and bi-microlocalization. The purpose of this paper is to establish the foundations of multi-microlocalization, in particular, to give the fiber formula for the multi-microlocalization functor and estimate of microsupport of a multi-microlocalized object. We briefly explain, in what follows, these two important results and their meanings.

The most fundamental question for the multi-microlocalization functor μ_χ along closed submanifolds $\chi = \{M_1, \dots, M_\ell\}$ is a shape of a cone on which a boundary value given by μ_χ is defined. The fiber formula gives us an explicit answer: A germ of $H^k(\mu_\chi(F))$ is isomorphic to local cohomology groups $\varinjlim_{G} H^k_G(F)$ where G is a vector sum of closed cones G_i's and each G_i is defined in the similar way as that in the fiber formula of the usual microlocalization functor along M_i. Therefore the multi-microlocalization functor can be understood as a natural extension of the usual microlocalization functor. Once we have grasped a geometrical aspect of multi-microlocalization, then the next fundamental problem to be considered is estimate of microsupport of $\mu_\chi(F)$ by that of F, for which the answer is quite simple and beautiful: The microsupport $SS(\mu_\chi(F))$ is contained in the multi-normal cone of $SS(F)$ along χ^*. Here χ^* is a family of Lagrangian submanifolds $\{T^*_{M_1}X, \ldots, T^*_{M_\ell}X\}$. This shows, in particular, soundness of our framework in the sense that the sharp estimate can be achieved by a geometrical tool (the multi-normal cone) already prepared in our framework. These two results have many applications, and some of them will be given in the last section of this paper.

The paper is organized as follows: We briefly recall, in Section 1, the theory of the multi-specialization developed in [3]. Then, in Section 2, we define the multi-microlocalization functor by repeatedly applying Sato's Fourier transformation to the multi-specialization functor. After showing several basic properties of the functor, we establish a fiber formula which explicitly describes a stalk of a multi-microlocalized object. In Section 3, after some geometrical preparations, we give an estimate of microsupport of a microlocalized object, that is our main result. Several applications of this result to $\mathscr{D}$-modules are studied in Section 4.

1. Multi-specialization: a review

In this section we recall some results of [3]. We first fix some notations, then we recall the notion of multi-normal deformation and the definition of the functor of multi-specialization with some basic properties.

1.1. Notations. — Let X be a real analytic manifold with $\dim X = n$, and let $\chi = \{M_1, \ldots, M_\ell\}$ be a family of closed submanifolds in X ($\ell \geqslant 1$). Throughout the paper all the manifolds are always assumed to be countable at infinity.

We set, for $N \in \chi$ and $p \in N$,

$$\mathrm{NR}_p(N) := \{M_j \in \chi;\ p \in M_j,\ N \not\subseteq M_j \text{ and } M_j \not\subseteq N\}.$$

Let us consider the following conditions for χ.

H1 Each $M_j \in \chi$ is connected and the submanifolds are mutually distinct, i.e., $M_j \neq M_{j'}$ for $j \neq j'$.

H2 For any $N \in \chi$ and $p \in N$ with $\mathrm{NR}_p(N) \neq \varnothing$, we have

$$\left(\bigcap_{M_j \in \mathrm{NR}_p(N)} T_pM_j\right) + T_pN = T_pX. \tag{1.1}$$

Note that, if χ satisfies the condition H2, the configuration of two submanifolds must be either 1. or 2. below.