Bulletin

de la SOCIÉTÉ MATHÉMATIQUE DE FRANCE

LOGARITHMIC BUNDLES OF DEFORMED WEYL ARRANGEMENTS OF TYPE A_2

Takuro Abe & Daniele Faenzi & Jean Vallès

Tome 144
Fascicule 4

2016

SOCIÉTÉ MATHÉMATIQUE DE FRANCE
Publié avec le concours du Centre national de la recherche scientifique
pages 745-761

Le *Bulletin de la Société Mathématique de France* est un périodique trimestriel de la Société Mathématique de France.

Fascicule 4, tome 144, décembre 2016

Diffusion

Maison de la SMF
Case 916 - Luminy
13288 Marseille Cedex 9
France
smf@smf.univ-mrs.fr

Hindustan Book Agency
O-131, The Shopping Mall
Arjun Marg, DLF Phase 1
Gurgaon 122002, Haryana
Inde

AMS
P.O. Box 6248
Providence RI 02940
USA
www.ams.org

Tarifs

Vente au numéro : 43 € ($ 64)
Abonnement Europe : 178 €, hors Europe : 194 € ($ 291)
Des conditions spéciales sont accordées aux membres de la SMF.

Secrétariat : Nathalie Christiaën

Bulletin de la Société Mathématique de France
Société Mathématique de France
Institut Henri Poincaré, 11, rue Pierre et Marie Curie
75231 Paris Cedex 05, France
Tél : (33) 01 44 27 67 99 • Fax : (33) 01 40 46 90 96
revues@smf.ens.fr • http://smf.emath.fr/

ISSN 0037-9484

Directeur de la publication : Stéphane SEURET

Bull. Soc. math. France
144 (4), 2016, p. 745–761

LOGARITHMIC BUNDLES OF DEFORMED WEYL ARRANGEMENTS OF TYPE A_2

by Takuro Abe, Daniele Faenzi & Jean Vallès

Abstract. — We consider deformations of the Weyl arrangement of type A_2, which include the extended Shi and Catalan arrangements. These last ones are well-known to be free. We study their sheaves of logarithmic vector fields in all other cases, and show that they are Steiner bundles. Also, we determine explicitly their unstable lines. As a corollary, some counter-examples to the shift isomorphism problem are given.

Résumé (*Fibrés logarithmiques des arrangements de Weyl déformés de type A_2*)
Nous considérons des déformations des arrangements de Weyl de type A_2, déformations dont les arrangements de Shi et de Catalan forment une classe particulière. Il est bien connu que ces derniers sont libres. Nous étudions les faisceaux de champs de vecteurs logarithmiques des autres arrangements déformés et montrons qu'ils sont des fibrés de Steiner. Nous déterminons explicitement leurs droites instables. Comme corollaire, des contres exemples du problème appelé « shift isomorphism » sont donnés.

Texte reçu le 25 décembre 2014, révisé le 30 mai 2016, accepté le 2 juin 2016.

Takuro Abe, Institute of Mathematics for Industry, Kyushu University, 744, Motooka, Nishi-Ku, Fukuoka 819-0395, Japan • *E-mail :* abe@imi.kyushu-u.ac.jp
Daniele Faenzi, Université de Bourgogne, Institut de Mathématiques de Bourgogne, UMR CNRS 5584, UFR Sciences et Techniques, Bâtiment Mirande, Bureau 310, 9 Avenue Alain Savary, BP 47870, 21078 Dijon Cedex, France • *E-mail :* daniele.faenzi@u-bourgogne.fr
Jean Vallès, Université de Pau et des Pays de l'Adour, Avenue de l'Université, BP 576, 64012 Pau Cedex, France • *E-mail :* jean.valles@univ-pau.fr

2010 Mathematics Subject Classification. — 52C35, 14F05, 32S22.

Key words and phrases. — Line arrangements, logarithmic sheaves, Weyl arrangements, root systems.

T. A. is supported by JSPS Grants-in-Aid for Young Scientists (B) No. 24740012. D. F. and J. V. partially supported by GEOLMI ANR-11-BS03-0011. All authors supported by Sakura Campus France project *Géométrie, combinatoire et topologie des arrangements d'hyperplans.*

Introduction

Let Φ be an irreducible crystallographic root system in Euclidean space $V \simeq \mathbb{R}^m$, let $\Phi^+ \subset \Phi$ be the positive roots, and let η be the Coxeter number of Φ. Let $x_1, \dots, x_m$ be coordinates of V, set $S = \mathbb{R}[x_0, \dots, x_m]$, and denote by $\mathrm{Der}(S)$ the free S-module of derivations of S, generated by the partial derivatives $\frac{\partial}{\partial x_0}, \dots, \frac{\partial}{\partial x_m}$. For $s \in \mathbb{Z}$ and $\alpha \in \Phi^+$, define the hyperplanes:

$$H_{\alpha,s} = \{x \in \mathbb{P}^m \mid \alpha(x_1, \dots, x_m) = s x_0\} \subset \mathbb{P}^m.$$

Fix integers $k, j \geq 0$, and define the (cone over the) *deformation of the Weyl arrangement of type* Φ:

$$\mathscr{A}_\Phi^{[-j,k+j]} = \{x_0 = 0\} \cup \{H_{\alpha,s} \mid \alpha \in \Phi^+, -j \leq s \leq k+j\}.$$

The combinatorics, topology and algebra of $\mathscr{A} = \mathscr{A}_\Phi^{[-j,k+j]}$ have been studied by several authors, for instance by Postnikov and Stanley in [10], by Athanasiadis in [3], by Edelman and Reiner in [5], and by Yoshinaga in [14], especially when $k \in \{0, 1\}$. In particular, the freeness of $\mathscr{A}$ when $k = 0, 1$ was conjectured by Edelman and Reiner and proved in [14] by Yoshinaga. By *freeness* here we mean freeness of logarithmic derivation module of $\mathscr{A}$:

$$D_0(\mathscr{A}) := \{\theta \in \mathrm{Der}(S) \mid \theta(f_{j,k}) = 0\},$$

where $f_{j,k}$ is the form of degree $n = |\mathscr{A}|$ given as product of linear forms defining the hyperplanes of $\mathscr{A}$. Equivalently, freeness means splitting of the the sheafification $T_{\mathscr{A}}$ of $D_0(\mathscr{A})$. This is a reflexive sheaf of rank m called *logarithmic sheaf*. It can also be defined as the kernel of the Jacobian map:

$$\mathscr{O}_{\mathbb{P}^m}^{m+1} \xrightarrow{\nabla(f_{j,k})} \mathscr{O}_{\mathbb{P}^m}(n-1).$$

In spite of the good knowledge of $T_{\mathscr{A}}$ for $k \in \{0, 1\}$, almost nothing is known about $T_{\mathscr{A}}$ for $k \geq 2$, not even for A_2. For example, setting $\mathscr{B} = \mathscr{A}_\Phi^{[-j-1,k+j+1]}$, the *shift isomorphism problem*, cf. [15, Remark 3.7] asks whether there is an isomorphism:

$$T_{\mathscr{A}} \simeq T_{\mathscr{B}}(\eta). \tag{1}$$

Another question is the *shifted dual isomorphism problem*, to the effect that:

$$T_{\mathscr{A}} \simeq T_{\mathscr{A}}^\vee(-\eta(k+2j+1)). \tag{2}$$

These isomorphisms hold when $k = 0, 1$ by [14]. However, even equality of characteristic polynomials (i.e., of Chern classes) of these sheaves is unknown in general: this is the so-called "functional equation" conjecture of [10], cf. also [15, Conjecture 3.4 and 3.5]. However the roots of the characteristic polynomial should have real part $\eta(k+2j+1)/2$ by the "Riemann hypothesis" of [10], verified for Φ of type A, B, C, D in [2].

In this paper, we are most interested in the case $\Phi = A_2$. We switch to the notation (z, x, y) rather than (x_0, x_1, x_2), and we fix $\mathscr{A} = \mathscr{A}_{A_2}^{[-j,k+j]}$. We have $\eta = 3$. In this case $T_{\mathscr{A}}$ is locally free (a vector bundle) of rank 2, and the lines of $\mathscr{A}$ are defined by vanishing of the form:

$$f_{j,k} = z \prod_{-j \le s \le k+j} (x - sz)(y - sz)(y + x - sz).$$

Concerning resolutions, our main theorem is the following.

THEOREM 1. — *For any $k \ge 2$ and $j \ge 0$, there is a resolution:*

$$0 \to \mathscr{O}_{\mathbb{P}^2}(-1)^{k-1} \to \mathscr{O}_{\mathbb{P}^2}^{k+1} \to T_{\mathscr{A}}(2k + 1 + 3j) \to 0.$$

In particular, $T_{\mathscr{A}}(2k + 1 + 3j)$ is a Steiner bundle.

By *Steiner bundle* here we mean a vector bundle whose resolution is given by a matrix of linear forms. This agrees and gives a new interpretation of the following formulas, easily obtained for instance counting multiple points and using [7, Remark 2.2]:

$$c_1(T_{\mathscr{A}}(2k + 3j + 1)) = k - 1, \qquad c_2(T_{\mathscr{A}}(2k + 3j + 1)) = \tfrac{k(k-1)}{2}.$$

Since $T_{\mathscr{A}}(2k + 1 + 3j)$ is a Steiner bundle, for any line $L \subset \mathbb{P}^2$, by restriction onto L we get a surjective map:

$$\mathscr{O}_L^{k+1} \longrightarrow T_{\mathscr{A}}(2k + 1 + 3j)|_L.$$

This implies that $T_{\mathscr{A}}(2k+1+3j)|_L = \mathscr{O}_L(a) \oplus \mathscr{O}_L(k-1-a)$ with $0 \le a \le k-1$. When $a = 0$ or $a = k - 1$ the number $|k - 1 - 2a|$ is as large as possible. This justifies the next definition, cf. [11, Page 508] or [6, Definition 2.1].

DEFINITION 1. — Let $k \ge 2$ and E be a Steiner bundle defined by:

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^2}(-1)^{k-1} \longrightarrow \mathscr{O}_{\mathbb{P}^2}^{k+1} \longrightarrow E \longrightarrow 0.$$

A line L such that $E|_L = \mathscr{O}_L \oplus \mathscr{O}_L(k - 1)$ or equivalently $H^0(\mathbb{P}^2, E^\vee|_L) \neq 0$ or equivalently $H^1(L, E|_L(-2)) \neq 0$ is called *unstable*. The set of such lines is denoted by $W(E)$, it is naturally a subscheme of $\check{\mathbb{P}}^2$. These unstable lines were first called *superjumping lines* in [4].

Our next result, tightly related with Theorem 1, deals with the set of unstable lines of $T_{\mathscr{A}}$, which we can determine explicitly. The figure shows them in case $j = 0$ and $k = 3$ or $k = 4$, the thick orange lines being unstable (the solid ones lie in the arrangement, the dashed ones don't).