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INCOMPRESSIBLE GRASSMANNIANS OF ISOTROPIC SUBSPACES

by Nikita A. Karpenko

Abstract. — We study 2-incompressible Grassmannians of isotropic subspaces of a quadratic form, of a hermitian form over a quadratic extension of the base field, and of a hermitian form over a quaternion algebra.

Résumé (*Grassmanniennes incompressibles d'espaces isotropes*)
Nous étudions des grassmanniennes 2-incompressibles de sous-espaces isotropes d'une forme quadratique ainsi que d'une forme hermitienne au-dessus d'une extension quadratique du corps de base ou au-dessus d'une algèbre de quaternions.

1. Introduction

Let F be a field and let p be a prime number. We refer to [6] and [15] for definitions and general discussion of canonical p-dimension and p-incompressibility. We only recall that canonical p-dimension $\operatorname{cdim}_p X$ of a smooth complete variety X is the least dimension of the image of a self-correspondence $X \rightsquigarrow X$

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of multiplicity prime to p; X is called p-incompressible, if $\operatorname{cdim}_p X = \dim X$, that is, if every self-correspondence of multiplicity prime to p is dominant.

We work with *projective homogeneous varieties*, i.e., twisted flag varieties under semisimple affine algebraic groups. One large class of such varieties, for which the p-incompressibility property is completely understood, is given by the generalized Severi-Brauer varieties of central simple algebras, [8]. In the present paper we study another large class which may be considered as the simplest one for which we do not possess a complete general criterion of p-incompressibility. Namely, we study Grassmannians of totally isotropic spaces of a fixed dimension for:

- a non-degenerate quadratic form (orthogonal case, algebraic groups of types $\mathscr{B}$ and $\mathscr{D}$), or
- a hermitian form over a separable quadratic extension field of F (unitary case, type ${}^2\!\mathscr{A}$), or
- a hermitian form over a quaternion division F-algebra (symplectic case, type $\mathscr{C}$), where in the characteristic 2 case the hermitian form is supposed to be *alternating*, [14, §4.A].

Note that in the symplectic case, only the Grassmannians of subspaces of *integral* dimension over the quaternion algebra are considered because the others are not interesting from the viewpoint of the canonical dimension. Also note that $p = 2$ is the only interesting prime for the varieties treated here.

Our main result on the orthogonal case is Theorem 3.1. Roughly speaking, it asserts that most of the necessary conditions of 2-incompressibility, established in [7], are actually necessary, see Remark 3.2. Therefore Theorem 3.1 may be considered as a step towards finding a precise criterion of 2-incompressibility for this type of varieties.

Our main results for the unitary and the symplectic case are Theorems 4.1 and 5.2. Although the varieties occurring in the three cases have quite different nature, we treat them in similar ways and the obtained results are parallel.

These results can be used to provide a conceptual proof of the criterion of p-incompressibility for products of two projective homogeneous varieties in the case where one of the varieties involved is a Grassmannian of isotropic subspaces, see [12]. Although the criterion has been recently proved in full in [11], the available general proof is very ad-hoc.

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2. Canonical p-dimension of a fibration

As shown in [10], canonical p-dimension of the product $X \times Y$ of projective homogeneous F-varieties X and Y has $\operatorname{cdim}_p X + \operatorname{cdim}_p Y_{F(X)}$ as an upper bound. We may view X as the base of the projection (a “trivial fibration”) $X \times Y \to X$, and $Y_{F(X)}$ is its generic fiber. In this section, we generalize this upper bound relation to the case of a more general fibration and also we sharpen the upper bound, replacing $\operatorname{cdim}_p X$ by an, in general, smaller integer $\operatorname{cdim}'_p X$ defined in terms of X and the function field of the total variety of the fibration.

Here is the type of the fibrations we are interested in. Let G be a quasi-split semisimple affine algebraic group over F becoming split over a finite extension field of a p-power degree, T a G-torsor over F, P a parabolic subgroup of G and P' a parabolic subgroup of G contained in P. We consider the fibration

$$\pi\colon Z := T/P' \to T/P =: X$$

of projective homogeneous varieties, and we write Y for its generic fiber. We are using the Chow group Ch with coefficients in $\mathbb{F}_p := \mathbb{Z}/p\mathbb{Z}$. In particular, the degree homomorphism deg on Ch_0 of a complete variety takes its values in $\mathbb{F}_p$.

Let us first recall

Proposition 2.1 ([7, Corollary 6.2]). — *Canonical p-dimension* $\operatorname{cdim}_p X$ *of X is the minimal integer d such that there exist a cycle class $\alpha \in \operatorname{Ch}^d X_{F(X)}$ and a cycle class $\beta \in \operatorname{Ch}_d X$ with* $\deg(\beta_{F(X)} \cdot \alpha) = 1 \in \mathbb{F}_p$.

Lemma 2.2. — *In the above settings, we have*

$$\operatorname{cdim}_p Z \leq \operatorname{cdim}'_p X + \operatorname{cdim}_p Y,$$

where $\operatorname{cdim}'_p X$ *is the minimal integer d such that there exist elements $\alpha \in \operatorname{Ch}^d(X_{F(Z)})$ and $\beta \in \operatorname{Ch}_d(X)$ with* $\deg(\alpha \cdot \beta_{F(Z)}) = 1$.

Remark 2.3. — Replacing $F(Z)$ by $F(X)$ in the definition of $\operatorname{cdim}'_p X$, we get $\operatorname{cdim}_p(X)$, see Proposition 2.1. Since $F(Z) \supset F(X)$, we have $\operatorname{cdim}'_p X \leq \operatorname{cdim}_p X$.

Remark 2.4. — In the case where the parabolic subgroup P' is special, Lemma 2.2 has been proved in [5, Lemma 5.3]. The proof was more complicated (and the statement—more specific) because Proposition 2.1 was not available at the time.