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COMPLEXES OF GROUPS AND GEOMETRIC SMALL CANCELATION OVER GRAPHS OF GROUPS

BY ALEXANDRE MARTIN

ABSTRACT. — We explain and generalize a construction due to Gromov to realize geometric small cancelation groups over graphs of groups as fundamental groups of non-positively curved 2-dimensional complexes of groups. We then give conditions so that the hyperbolicity and some finiteness properties of the small cancelation quotient can be deduced from analogous properties for the local groups of the initial graph of groups.

RÉSUMÉ (*Complexes de groupes et petite simplification géométrique sur les graphes de groupes*). — Nous généralisons une construction de Gromov afin de réaliser certains groupes à petite simplification géométrique sur un graphe de groupes comme groupes fondamentaux de complexes de groupes de dimension 2 à courbure négative ou nulle. Nous donnons ensuite des conditions pour que l'hyperbolicité et certaines propriétés de finitude de tels groupes se déduisent des propriétés analogues pour les groupes locaux du graphe de groupes initial.

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1. Introduction and statement of results

Small cancelation theory deals with the following problem: given a group G and a family $(H_i)_{i\in I}$ of subgroups, find conditions under which one understands the quotient $G/\ll H_i \gg$ (where $\ll H_i \gg$ denotes the normal closure of the subgroup generated by the H_i).

In classical small cancelation theory, G is a finitely generated free group and each H_i is an infinite cyclic subgroup generated by a cyclically reduced element. Small cancelation conditions essentially ask that the length of a common subword of two relators be short relatively to the length of the relators. Such conditions come in various flavors and the overall theory has been generalized to many settings, such as small cancelation over graphs of groups [16] or small cancelation over a hyperbolic group [21, 10, 6]. In [13], Gromov gave a geometric version of small cancelation, using the language of *rotation families* (see Definition 4.1). In this case, the group G acts isometrically on a hyperbolic metric space X and each subgroup H_i stabilizes a given subspace Y_i. Small cancelation conditions in this context ask that the overlap between two such subspaces be small with respect to the injectivity radii of the H_i. This point of view was used for instance in [8, 11, 9].

Small cancelation theory offers powerful tools to study various classes of groups, and provide many examples of groups with exotic properties [13].

Some small cancelation conditions have strong geometric consequences. In [12], Gromov proved that groups satisfying the so-called geometric small cancelation condition $C''(1/6)$ (see Definition 4.2) act properly and cocompactly on a CAT(0) space. The first goal of this article is to detail this construction and to extend it to the case of small cancelation over a graph of groups; in the case of classical small cancelation, this construction was explained by Vinet (unpublished). More precisely, we prove the following:

THEOREM 1.1. — *Let $G(\Gamma)$ be a graph of groups over a finite simplicial graph Γ, with fundamental group G and Bass–Serre tree T. Let $(A_\xi, H_\xi)_{\xi\in\Xi}$ be a rotation family such that the following condition holds:*

(∗) *For every $\xi \in \Xi$, H_ξ is an infinite cyclic subgroup generated by a hyperbolic element with axis A_ξ. Moreover, the global stabilizer of A_ξ is virtually cyclic, and every element of the global stabilizer of A_ξ fixes pointwise the endpoints of A_ξ.*

If $(A_\xi, H_\xi)_{\xi\in\Xi}$ satisfies the geometric small cancelation condition $C''(1/6)$, then the quotient $G/\ll H_\xi \gg$ is the fundamental group of a non-positively curved 2-dimensional complex of groups, the local groups of which are either finite or subgroups of the local groups of $G(\Gamma)$.

In [12], Gromov even constructed actions of $C''(1/6)$ small cancelation groups on CAT(κ) spaces for some $\kappa < 0$. While it would be possible to adapt the previous constructions to obtain such actions, we restrict to actions on (piecewise Euclidean) CAT(0) complexes for two reasons. First, the actions considered here are non-proper, so a negatively-curved assumption on the space would not translate immediately to a property of the quotient group (note however that we will consider the hyperbolicity of such a quotient in Section 7). More importantly, having an action on a piecewise-Euclidean complex (or more generally on a M_κ-complex in the sense of Bridson [4]) will be used in Section 7 to study the geometry of the quotient group.

Let us give a few details about the construction. The family of axes A_ξ is used to construct the so-called coned-off space $\widehat{T}$ (see Definition 5.2) and the quotient space $\ll H_\xi \gg \backslash \widehat{T}$ comes with an action of $G/_{\ll H_\xi \gg}$. Since two axes $A_\xi, A_{\xi'}$ may share more than one edge, the space $\ll H_\xi \gg \backslash \widehat{T}$ does not have a CAT(0) geometry in general. Generalising an idea of Gromov [12], we want to construct a CAT(0) complex with a cocompact action of $G/_{\ll H_\xi \gg}$ by identifying certain portions of $\ll H_\xi \gg \backslash \widehat{T}$. Understanding the resulting action, and in particular the various stabilizers, turns out to be a non-trivial problem. To avoid this issue, we use a different approach, using the theory of complexes of group. We start by considering the complex of groups one would expect from the action of $G/_{\ll H_\xi \gg}$ on the space obtained from $\ll H_\xi \gg \backslash \widehat{T}$ after performing Gromov's construction. This complex of groups decomposes as a tree of complexes of groups (see Section 3 for the definition), where the various pieces are much easier to handle. In particular, our approach allows us to consider only the action of G on $\widehat{T}$, instead of dealing with the action of $G/_{\ll H_\xi \gg}$ on $\ll H_\xi \gg \backslash \widehat{T}$. Using standard results on complexes of groups, we prove that this complex of groups is indeed non-positively curved and has $G/_{\ll H_\xi \gg}$ as fundamental group.

Note that our approach can be used to provide geometric proofs of several facts which are well known for classical small cancelation theory. For instance, it is well known that for classical small cancelation theory over amalgamated products or HNN extensions, the quotient map $G \to G/_{\ll H_\xi \gg}$ embeds each local group of G [16, Theorems V.11.2 and V.11.6]. In this article, and under the assumptions of Theorem 1.1, this is a direct consequence of the developability of the associated complex of groups (see Corollary 5.17).

In Theorem 1.1, we make the assumption $(*)$ that every element in the (global) stabilizer of an axis of the form A_ξ fixes its endpoints. This condition is required to adapt the construction of Gromov to this more general setting. Note that it is automatically satisfied for classical small cancelation theory, that is, for the action of the free group F_n on the $2n$-valent tree. Note that condition $(*)$ is equivalent to requiring that: