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## GROUP SCHEMES WITH $\mathbb{F}_q$ -ACTION

BY THOMAS POGUNTKE

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ABSTRACT. — Via a construction due to V. Drinfel'd, we prove an equivalence of categories, generalizing the equivalence between commutative flat group schemes in characteristic  $p$  with trivial Verschiebung and their Dieudonné modules to group schemes with  $\mathbb{F}_q$ -action.

RÉSUMÉ (*Schémas en groupes avec  $\mathbb{F}_q$ -action*). — Au moyen d'une construction par V. Drinfel'd, nous prouvons une équivalence de catégories, généralisant l'équivalence entre les schémas en groupes plats commutatifs en caractéristique  $p$  annulé par le décalage et leurs  $\mathbb{F}_p$ -modules de Dieudonné aux schémas en groupes avec action de  $\mathbb{F}_q$ .

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### 1. Introduction

Let  $p$  be a prime and let  $k$  be a field of characteristic  $p$ . Denote by  $\text{Gr}_k^+$  the category of affine commutative group schemes over  $k$  which can be embedded into  $\mathbb{G}_a^N$  for some set  $N$ . We assign to  $G \in \text{Gr}_k^+$  its Dieudonné  $\mathbb{F}_p$ -module  $\mathcal{M}(G) = \text{Hom}_{\text{Gr}_k^+}(G, \mathbb{G}_a)$ , with the obvious left module structure over  $\text{End}_{\text{Gr}_k^+}(\mathbb{G}_a) \cong k[F]$ , the non-commutative polynomial ring with

$$F\lambda = \lambda^p F \text{ for } \lambda \in k.$$

These Dieudonné modules completely classify group schemes of the above type, as shown by the following theorem.

**THEOREM 1.1** ([2], IV, §3, 6.7). — *The contravariant functor  $\mathcal{M}$  defines an exact anti-equivalence of categories*

$$(1.1) \quad \mathcal{M} : \text{Gr}_k^+ \longrightarrow k[F]\text{-Mod.}$$

*Under this duality, group schemes of finite presentation correspond to finitely generated  $k[F]$ -modules, and finite group schemes to finite-dimensional  $k$ -vector spaces.*

The above result allows us to describe the structure of our category over a perfect field, and its simple objects if  $k$  is algebraically closed.

**THEOREM 1.2** ([2], IV, §3, 6.9). — *Let  $k$  be a perfect field. Then  $G \in \text{Gr}_k^+$  is algebraic if and only if it can be written as a product*

$$G \cong \mathbb{G}_a^n \times \pi_0(G) \times H,$$

*where  $n \in \mathbb{N}$ ,  $H$  is a finite product of group schemes of the form  $\alpha_{p^s}$ , and  $\pi_0(G)$  is an étale sheaf of finite  $\mathbb{F}_p$ -vector spaces. If  $k$  is algebraically closed, then*

$$\pi_0(G) \cong (\mathbb{F}_p)^m, m \in \mathbb{N}.$$

On the other hand, let  $S$  be a scheme of characteristic  $p$ . Consider the category  $\text{gr}_S^{+\vee}$  of flat group schemes locally of finite presentation over  $S$  of height  $\leq 1$  (i.e., killed by their Frobenius). Let  $p\text{-Lie}_S$  denote the category of finite locally free  $\mathcal{O}_S$ - $p$ -Lie algebras. Then we have the following classification theorem.

**THEOREM 1.3** ([7], Remark 7.5). — *The covariant functor*

$$\mathcal{L} : \text{gr}_S^{+\vee} \longrightarrow p\text{-Lie}_S, G \longmapsto \text{Lie}(G),$$

*defines an equivalence of categories.*

Two of our main results generalize Theorem 1.1, and reduce (for “ $q = p$ ”) to Theorem 1.3 via Cartier duality, respectively. Moreover, we formulate two conjectures under which they unify.

Assume that  $S$  is an  $\mathbb{F}_q$ -scheme for some prime power  $q = p^r$ . Our group schemes  $G$  are affine, commutative, flat over  $S$  and carry an  $\mathbb{F}_q$ -action. We require that locally on  $S$ , there is an embedding  $G \hookrightarrow \mathbb{G}_a^N$  for some set  $N$ , which respects the  $\mathbb{F}_q$ -actions.

The category of these group schemes will be denoted by  $\mathbb{F}_q\text{-Gr}_S^+$ , and its full subcategory of finite group schemes of finite presentation is called  $\mathbb{F}_q\text{-gr}_S^+$ .

On the other hand, we consider left  $\mathcal{O}_S[F^r]$ -modules, which are flat as  $\mathcal{O}_S$ -modules. They are called  $\mathbb{F}_q$ -shtukas over  $S$ , and their category is denoted by  $\mathbb{F}_q\text{-Sht}_S$ . We write  $\mathbb{F}_q\text{-sht}_S$  for the full subcategory of  $\mathbb{F}_q\text{-Sht}_S$  of locally free modules of finite rank over  $\mathcal{O}_S$ .

We study the following generalization of the contravariant functor (1.1),

$$\mathcal{M}_q = \mathcal{M} : \mathbb{F}_q\text{-gr}_S^+ \longrightarrow \mathbb{F}_q\text{-sht}_S, G \longmapsto \text{Hom}_{\mathbb{F}_q\text{-Gr}_S^+}(G, \mathbb{G}_a).$$

We also explain the construction of a functor in the other direction,

$$\mathcal{G}_q = \mathcal{G} : \mathbb{F}_q\text{-sht}_S \longrightarrow \mathbb{F}_q\text{-gr}_S^+,$$

which is fully faithful and left-adjoint to  $\mathcal{M}$ . However,  $\mathcal{G}_q$  does not define an equivalence of categories for  $q \neq p$ . Rather, we describe a full subcategory  $\mathbb{F}_q\text{-gr}_S^{+,b}$  of *balanced* group schemes in  $\mathbb{F}_q\text{-gr}_S^+$ , and prove that it is the essential image of  $\mathcal{G}$ .

Namely, let  $G = \text{Spec}(B_G) \in \mathbb{F}_q\text{-gr}_S^+$ . We show that the space of primitive elements in  $B_G$  decomposes into eigenspaces for the  $\mathbb{F}_q^\times$ -action as

$$(1.2) \quad \text{Prim}(B_G) = \bigoplus_{s=0}^{r-1} \text{Prim}_{p^s}(B_G).$$

We call  $G$  balanced, if the  $p$ -Frobenii  $\text{Prim}_{p^t}(B_G) \rightarrow \text{Prim}_{p^{t+1}}(B_G), x \mapsto x^p$ , are bijective for all  $0 \leq t < r-1$ . Note that when  $q = p$ , we recover  $\mathbb{F}_p\text{-gr}_S^{+,b} = \text{gr}_S^+$ .

**THEOREM 1.4.** — *The functor  $\mathcal{G} : \mathbb{F}_q\text{-sht}_S \rightarrow \mathbb{F}_q\text{-gr}_S^{+,b}$  defines an exact anti-equivalence of categories with quasi-inverse  $\mathcal{M}$ .*

Our definition of the balanced subcategory of  $\mathbb{F}_q\text{-gr}_S^+$  is inspired by Raynaud’s paper [15]. He considers finite commutative group schemes  $G$  with an action of  $\mathbb{F}_q$ , and the decomposition of the augmentation ideal into eigenspaces for the  $\mathbb{F}_q^\times$ -action,

$$I_G = \bigoplus_{j=1}^{q-1} I_j,$$

similarly to (1.2). Note that all summands  $I_j$  are finite locally free  $\mathcal{O}_S$ -modules. Raynaud imposes the condition that  $\text{rk}(I_j) = 1$ , for all  $j$ .

We define a group scheme  $G \in \mathbb{F}_q\text{-gr}_S^+$  to be *quasi-balanced* if  $\text{rk}(I_j)$  is the same for all  $j$ . This turns out to be almost the same as being balanced; in