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DECODING RAUZY INDUCTION: AN ANSWER TO BUFETOV'S GENERAL QUESTION

by Jon Fickenscher

Abstract. — Given a typical interval exchange transformation, we may naturally associate to it an infinite sequence of matrices through Rauzy induction. These matrices encode visitations of the induced interval exchange transformations within the original. In 2010, W. A. Veech showed that these matrices suffice to recover the original interval exchange transformation, unique up to topological conjugacy, answering a question of A. Bufetov. In this work, we show that interval exchange transformation may be recovered and is unique modulo conjugacy when we instead only know consecutive products of these matrices. This answers another question of A. Bufetov. We also extend this result to any inductive scheme that produces square visitation matrices.

Résumé (*Décoder l'induction de Rauzy: Une réponse à la question générale de Bufetov*). — Etant donné une transformation d'échange d'intervalles typique, nous pouvons y associer naturellement une séquence infinie de matrices via l'induction de Rauzy. Ces matrices encodent les visites des transformations d'échanges d'intervalles induites dans l'intervalle original. En 2010, W. A. Veech a montré que ces matrices suffisent pour retrouver la transformation d'échange d'intervalles originale, unique à conjugaison topologique près, rèpondant à une question de A. Bufetov. Dans ce travail, nous montrons que la transformation d'échange d'intervalles peut être retrouvée et est unique à conjugaison près lorsque l'on connaît plutôt des produits consécutifs de ces matrices. Ceci répond à une autre question de A. Bufetov. Nous étendons également ce résultat à tout schéma inductif qui produit des matrices de visite carrées.

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1. Introduction

Interval exchange transformations (IET's) are invertible piece-wise translations on an interval I. They are typically defined by a permutation π on $\{1,\ldots,n\}$ and a choice of partitioning of I into sub-intervals $I_1,\ldots,I_n$ with respective lengths $\lambda_1,\ldots,\lambda_n$. The sub-intervals are reordered by T according to π.

For almost every IET T (i.e., for every appropriate π and Lebesgue almost every $\lambda=(\lambda_1,\ldots,\lambda_n)\in\mathbb{R}^n_+$), Rauzy induction, as defined in [6], is a map that sends an IET T on I to its first return T' on $I'\subset I$ for suitably chosen I'. For almost every(1) IET T, Rauzy induction may be applied infinitely often. This yields a sequence $T^{(k)}$, $k\geq 0$, of IET's so that each transition $T^{(k-1)}\mapsto T^{(k)}$ is the result of a Rauzy induction. To each step we may define a *visitation matrix* A_k so that $(A_k)_{ij}$ counts the number of disjoint images of the intervals $I^{(k)}_j$ in $I^{(k-1)}_i$ before return to $I^{(k)}$. It is part of the general theory of IET's that the initial π and the sequence of A_k's define T uniquely up to topological conjugacy. In preparation for [1], A. Bufetov posed the following.

QUESTION 1 (A. Bufetov). — *Given only the sequence of A_k's, can the initial permutation π be determined and is it unique?*

In response, W. A. Veech gave an affirmative answer in [10, Theorem 1.2]. This allowed A. Bufetov to ensure the injectivity of a map that intertwines the Kontsevich-Zorich cocycle with a renormalization cocycle (see the remark ending Section 4.3.1 in [1]).

However, if another induction scheme was used to get visitation matrices, we may not know each individual A_k. For instance, we may follow A. Zorich's acceleration of Rauzy induction (see [14]) or choose to induce on the first interval I_1. In either of these cases, our visitation matrix B will actually be a product $A_1\cdots A_N$ of the A_k's realized by Rauzy induction. Motivated by this, we say that a sequence B_ℓ, $\ell\in\mathbb{N}$, is a product of the A_k's if there exist an increasing sequence of integers k_ℓ, $\ell\geq 0$, so that $k_0=0$ and $B_\ell=A_{k_{\ell-1}+1}A_{k_{\ell-1}+2}\cdots A_{k_\ell}$ for each $\ell\geq 1$. We now are able to pose Bufetov's second, more general, question.

QUESTION 2 (A. Bufetov). — *Given instead a sequence B_ℓ, $\ell\in\mathbb{N}$, of products of the A_k's, can the initial permutation π still be determined and is it unique?*

This work is dedicated to answering this second question and its generalizations. We answer in the affirmative by our main results. Extended Rauzy induction is more general than regular Rauzy induction and is discussed in Section 2.5 before Lemma 2.15.

1. For every appropriate π and Lebesgue almost every $\lambda=(\lambda_1,\ldots,\lambda_n)\in\mathbb{R}^n_+$.

Main Theorem 1. — *If $B_1, B_2, B_3, \ldots$ are consecutive matrix products defined by an infinite sequence of steps of (extended) Rauzy induction, then the initial permutation π is unique.*

Recently, J. Jenkins proved this result in [3] for the 3×3 matrix case. He then explored the 4×4 case numerically.

In the most general case, we call the inductions on $I' \supsetneq I'' \supsetneq I'' \supsetneq \cdots$ an *admissible induction sequence* if the $n \times n$ visitation matrices A_k from $T^{(k-1)}$ to $T^{(k)}$ are well-defined. We then are able to answer Bufetov's question in a much broader setting.

Main Theorem 2. — *If visitation matrices $B_1, B_2, B_3, \ldots$ are defined by an admissible induction sequence, then the initial permutation π is unique.*

Outline of Paper. — In Section 2 we establish our notation and provide known results concerning IET's and related objects as well as general linear algebra. In particular, the anti-symmetric matrix L_π is defined given π, and this matrix plays a central role here. In Section 3 the Perron-Frobenius eigenvalue and eigenvector are discussed. The main argument of that section is Corollary 3.5, which says that the Perron-Frobenius eigenvector cannot be in the nullspace of any linear combination $L_\pi - cL_{\pi'}$ for permutations π, π' and scalar c. Section 4 begins with a reduction of Main Theorem 1 to a special case, stated as the Main Lemma. The section ends with a proof of the Main Lemma. Section 5 reduces Main Theorem 2 to Main Theorem 1 by Lemma 5.1. This lemma states that any admissible induction sequence must arise from extended Rauzy induction. Appendix A provides further results concerning admissibility and induced maps which lead to the proof of Lemma 5.1 in Appendix B.

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2. Definitions

An *interval* or *sub-interval* is of the form $[a, b)$ for $a < b$, i.e., a non-empty subset of $\mathbb{R}$ that is closed on the left and open on the right. If $I = [a, b)$ is an interval, $|I| = b - a$ denotes its *length*. For a set C, $\#C$ denotes its cardinality. A *translation* $\phi : I \to J$ for intervals I and J is any function that may be expressed as $\phi(x) = x + c$ for constant c. If $\psi : C \to D$ is a function and $E \subseteq C$ we use the notation ψE to mean the *image of E by ψ*, or $\psi E = \{\psi(c) : c \in E\} \subseteq D$. For $\lambda \in \mathbb{R}^n_+$, or a vector in $\mathbb{R}^n$ with all positive entries, $|\lambda| = \lambda_1 + \cdots + \lambda_n$ denotes the 1-norm of λ.