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A PARADIFFERENTIAL REDUCTION FOR THE GRAVITY-CAPILLARY WAVES SYSTEM AT LOW REGULARITY AND APPLICATIONS

by Thibault de Poyferré & Quang-Huy Nguyen

Abstract. — We consider in this article the system of gravity-capillary waves in all dimensions and under the Zakharov/Craig-Sulem formulation. Using a paradifferential approach introduced by Alazard-Burq-Zuily, we symmetrize this system into a quasilinear dispersive equation whose principal part is of order $\frac{3}{2}$. The main novelty, compared to earlier studies, is that this reduction is performed at the Sobolev regularity of quasilinear pdes: $H^s(\mathbf{R}^d)$ with $s > \frac{3}{2} + \frac{d}{2}$, d being the dimension of the free surface.

From this reduction, we deduce a blow-up criterion involving solely the Lipschitz norm of the velocity trace and the $C^{\frac{5}{2}+}$-norm of the free surface. Moreover, we obtain an a priori estimate in the H^s-norm and the contraction of the solution map in the $H^{s-\frac{3}{2}}$-norm using the control of a Strichartz norm. These results have been applied in establishing a local well-posedness theory for non-Lipschitz initial velocity in our companion paper [24].

Résumé (*Une réduction paradifférentielle du système des vagues de gravité-capillarité à basse régularité et applications*). — Dans cet article, nous étudions le système des vagues de gravité-capillarité en toutes dimensions, dans la formulation de Zakharov, Craig et Sulem. À l'aide d'une approche paradifférentielle introduite par Alazard, Burq et Zuily, nous symétrisons ce système en une équation dispersive quasilinéaire dont

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le terme principal est d'ordre $\frac{3}{2}$. La principale nouveauté par rapport aux études précédentes est que cette réduction est effectuée au niveau de régularité des EDPs quasilinéaires : $H^s(\mathbf{R}^d)$ avec $s > \frac{3}{2} + \frac{d}{2}$, d étant la dimension de la surface libre. À partir de cette réduction, nous déduisons un critère d'explosion n'impliquant que la norme Lipschitz de la trace de la vitesse et la norme $C^{\frac{5}{2}+}$ de la surface libre. En outre, nous obtenons une estimation a priori de la norme H^s et la contraction de l'application solution dans la norme $H^{s-\frac{3}{2}}$, en utilisant le contrôle d'une norme de Strichartz. Ces résultats ont été utilisés pour développer une théorie de Cauchy locale pour des vitesses initiales non Lipschitz, dans le papier compagnon [24].

1. Introduction

We consider the system of gravity-capillary waves describing the motion of a fluid interface under the effect of both gravity and surface tension. From the well-posedness result in Sobolev spaces of Yosihara [33] (see also Wu [31, 32] for pure gravity waves) it is known that the system is quasilinear in nature. In the more recent work [2], Alazard-Burq-Zuily showed explicitly this quasilinearity by using a paradifferential approach (see Appendix 6) to symmetrize the system into the following paradifferential equation

$$\big(\partial_t + T_{V(t,x)} \cdot \nabla + iT_{\gamma(t,x,\xi)}\big)u(t,x) = f(t,x) \tag{1.1}$$

where V is the horizontal component of the trace of the velocity field on the free surface, γ is an elliptic symbol of order $3/2$, depending only on the free surface. In other words, the transport part comes from the fluid and the dispersive part comes from the free boundary. The reduction (1.1) was implemented for

$$u \in L^\infty_t H^s_x \quad s > 2 + \frac{d}{2}, \tag{1.2}$$

d being the dimension of the free surface. It has many consequences, among them are the local well-posedness and smoothing effect in [2], Strichartz estimates in [3]. As remarked in [2], $s > 2 + d/2$ is the minimal Sobolev index (in term of Sobolev's embedding) to ensure that the velocity filed is Lipschitz up to the boundary, without taking into account the dispersive property. From the works of Alazard-Burq-Zuily [4, 1], Hunter-Ifrim-Tataru [14] for pure gravity waves, it seems natural to require that *the velocity is Lipschitz* so that the particles flow is well-defined, in view of the Cauchy-Lipschitz theorem. On the other hand, from *the standard theory of quasilinear pdes*, it is natural to ask if the reduction (1.1) holds at the Sobolev threshold $s > 3/2 + d/2$ and then, if a local-wellposedness theory holds at the same level of regularity? The two observations above motivate us to study the gravity-capillary system at the following regularity level:

$$u \in \mathcal{X} := L^\infty_t H^s_x \cap L^p_t W^{2,\infty}_x \quad \text{with } s > \frac{3}{2} + \frac{d}{2}, \tag{1.3}$$

which exhibits a gap of 1/2 derivative that may be filled up by Strichartz estimates. (1.13) means that on the one hand, the Sobolev regularity is that of quasilinear equations of order 3/2; on the other hand, the $L^p_t W^{2,\infty}_x$-norm ensures that the velocity is still Lipschitz for *a.e.* $t \in [0,T]$ (which is the threshold (1.2) after applying Sobolev's embedding).

By sharpening the analysis in [2], we shall perform the reduction (1.1) assuming merely the regularity $\mathcal{X}$ of the solution. In order to do so, the main difficulty, compared to [2], is that further studies of the Dirichlet-Neumann operator in Besov spaces are demanded. Moreover, we have to keep all the estimates in the analysis to be *tame*, i.e., linear with respect to the highest norm which is the Hölder norm in this case.

From this reduction, we deduce several consequences. The first one will be an *a priori estimate for the Sobolev norm* $L^\infty_t H^s_x$ using in addition the Strichartz norm $L^p_t W^{2,\infty}_x$ (see Theorem 1.1 below for an exact statement). This is an expected result, which follows the pattern established for other quasilinear equations. However, for water waves, it requires much more care due to the fact that the system is nonlocal and highly nonlinear. This problem has been addressed by Alazard-Burq-Zuily [1] for pure gravity water waves. In the case with surface tension, though the regularity level is higher, it requires a more precise analysis of the Dirichlet-Neumann operator in that lower order terms in the expansion of this operator need to be taken into consideration (se Proposition 3.6 below).

Another consequence will be a *blow-up criterion* (see Theorem 1.3), which implies that the solution can be continued as long as the $\mathcal{X}$-norm of u remained bounded (at least in the infinite depth case) with $p = 1$, i.e., merely integrable in time. It also implies that, starting from a smooth datum, the solution remains smooth provided its C^{2+}-norm is bounded in time.

For more precise discussions, let us recall the Zakharov/Craig-Sulem formulation of water waves.

1.1. The Zakharov/Craig-Sulem formulation. — We consider an incompressible, irrotational, inviscid fluid with unit density moving in a time-dependent domain

$$\Omega = \{(t,x,y) \in [0,T] \times \mathbf{R}^d \times \mathbf{R} : (x,y) \in \Omega_t\}$$

where each Ω_t is a domain located underneath a free surface

$$\Sigma_t = \{(x,y) \in \mathbf{R}^d \times \mathbf{R} : y = \eta(t,x)\}$$

and above a fixed bottom $\Gamma = \partial\Omega_t \setminus \Sigma_t$. We make the following separation assumption (H_t) on the domain at time t:
Ω_t is the intersection of the half space

$$\Omega_{1,t} = \{(x,y) \in \mathbf{R}^d \times \mathbf{R} : y < \eta(t,x)\}$$