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Société Mathématique de France
Institut Henri Poincaré, 11, rue Pierre et Marie Curie
75231 Paris Cedex 05, France
Tél : (33) 1 44 27 67 99 • Fax : (33) 1 40 46 90 96
bulletin@smf.emath.fr • smf.emath.fr

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CENTRAL POINTS OF THE DOUBLE HEPTAGON TRANSLATION SURFACE ARE NOT CONNECTION POINTS

by Julien Boulanger

Abstract. — We consider flow directions on the translation surfaces formed from double $(2n+1)$-gons and give a sufficient condition in terms of a natural continued fractions algorithm for a direction to be hyperbolic in the sense that it is a fixed direction for some hyperbolic element of the Veech group of the surface. In particular, we give explicit points with coordinates in the trace field of the double heptagon translation surface, that are not so-called connection points. Among these are the central points of the heptagons, giving a negative answer to a question by P. Hubert and T. Schmidt [1].

Résumé (*Les points centraux du double heptagone ne sont pas des points de connexion*). — On s'intéresse au flot directionnel sur les surfaces de translation obtenues à partir de deux $(2n+1)$-gones dont on a recollé les côtés parallèles, et on donne une condition suffisante pour qu'une direction soit hyperbolique, c'est à dire fixée par une direction hyperbolique du groupe de Veech, en termes d'un algorithme de fractions continues naturel sur les directions de la surface. En particulier, cela nous permet d'exhiber des points sur le double heptagone à coordonnées dans le corps de trace qui ne sont pas des points de connexion. Parmi ces points on peut notamment trouver les points centraux des heptagones, ce qui donne une réponse négative à une question de P. Hubert et T. Schmidt [1].

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Julien Boulanger, Institut Fourier, UMR 5582, Laboratoire de Mathématiques. Université Grenoble Alpes, CS 40700, 38058 Grenoble cedex 9, France • *E-mail :* julien.boulanger@univ-grenoble-alpes.fr

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1. Introduction and statement of the results

A translation surface is a genus g topological surface with an atlas of charts on the surface minus a finite set of points such that all transition functions are translations. These surfaces can also be described as the surfaces obtained by gluing pairs of opposite parallel sides of a collection of Euclidean polygons by translations. Such surfaces arise naturally in the study of billiard table dynamics: the Katok–Zemlyakov unfolding procedure, which consists in reflecting the billiard every time the trajectory hits an edge instead of reflecting the trajectory, replaces the billiard flow on a polygon by a directional flow on isometric translation surfaces. The study of translation surfaces has been flourishing, with major recent advances such as the results in [12], [10], or [11], but there still remains various open questions, for instance in the area of Veech groups. One of these questions is to characterize so-called connection points, for which little is known for translation surfaces whose trace field is of degree 3 or more over $\mathbb{Q}$. In this paper, we look at two particular points of the double heptagon surface, whose trace field is cubic over $\mathbb{Q}$, and show that they are not connection points. For surveys about translation surfaces, see [25] and [24], and for Veech groups, see [16].

Before looking at connection points, one needs to understand better parabolic (or hyperbolic) directions; that is, directions fixed by a parabolic (or hyperbolic) element of the Veech group. For Veech surfaces, periodic directions, saddle connection directions and directions fixed by parabolic elements of the Veech group coincide. For these terms, see the background and [16]. For translation surfaces whose trace field is quadratic or $\mathbb{Q}$, C. McMullen showed in [18] that (after a natural normalization) the periodic directions are exactly those with slopes in the trace field. When the trace field is of higher degree, it is no longer true, and the periodic directions in general form a proper subset of the directions whose slope belong to the trace field. D. Davis and S. Lelièvre [8] characterized the parabolic directions for the double pentagon surface using a continued fractions algorithm. Their results can be directly extended to the $(2n+1)$-gon, which has a trace field of degree n over $\mathbb{Q}$.

In this paper, we use the algorithm to characterize hyperbolic directions whose slopes belong to the trace field for each double $(2n+1)$-gon surface, which are made of two copies of a $(2n+1)$-gon with parallel opposite sides glued together. We find explicit examples of such directions for the double heptagon. This allows us to prove that central points of the double heptagon are not connection points, see Theorem 1.3. This answers negatively a question of P. Hubert and T. Schmidt. Recall that the central points of the double heptagon are the centers of the heptagons. A nonsingular point of a translation surface is called a connection point if every separatrix passing through this point can be extended to a saddle connection. In fact, the author does not know any example of a nonperiodic connection point[1] for a translation surface whose trace field is of degree 3 over $\mathbb{Q}$ or higher.

THEOREM 1.1. — *Let $n \geq 2$, for the double $(2n+1)$-gon surface, the directions that end in a periodic sequence (of period ≥ 2) for the continued fractions algorithm are hyperbolic directions.*

PROPOSITION 1.2 (Double heptagon case). — *For the double heptagon surface, there are hyperbolic directions in the trace field.*

This proposition is already known from [2] and [13], where a different method is used. Our method provides an answer to the question of central points as connection points, which was not known.

THEOREM 1.3. — *Central points of the double heptagon are not connection points.*

Moreover, one can look at double $(2n+1)$-gons with more sides. For example, the same result holds for the double nonagon:

THEOREM 1.4. — *Central points of the double nonagon are not connection points.*

Moreover, various tests that we conducted suggest the following conjecture, which is not new since we found the same ideas in [13].

CONJECTURE 1.5. — *For the double heptagon and the double nonagon, all the directions in the trace field are either parabolic or hyperbolic.*

What is interesting is that these results do not seem to generalize to the double hendecagon, for example. In fact, for the double hendecagon, we were not able to find any direction in the trace field that ends in a periodic sequence. These issues will be discussed in Section 5.

2. Background

A *translation surface* (X, ω) is a real compact genus g surface X with an atlas ω such that all transition functions are translations except on a finite set of singularities Σ, along with a distinguished direction. Alternatively, it can be seen as a surface obtained from a finite collection of polygons embedded in $\mathbb{C}$ by gluing pairs of parallel opposite sides by translation. We get a surface X with a flat metric and a finite number of singularities. We define $X' = X - \Sigma$, which inherits the translation structure of X and defines a Riemannian structure on X'. Therefore, we have notions of geodesics, length, angle, and geodesic

1. A point is *periodic* if its orbit under the action of the affine group is finite, otherwise it is nonperiodic, see [15].

flow (called directional flow). This allows us make the following definitions, which will be useful in Section 4.

DEFINITIONS 2.1. — (i) A *separatrix* is a geodesic line emanating from a singularity.

(ii) A *saddle connection* is a separatrix connecting singularities without any singularities on its interior.

(iii) A nonsingular point of the translation surface is called a *connection point*, if every separatrix passing through this point can be extended to a saddle connection.

The action of $GL_2^+(\mathbb{R})$ on polygons induces an action on the moduli space of translation surfaces (see, for example, [25]). Two surfaces are affinely equivalent, if they lie in the same orbit. The stabilizer of a given translation surface X is called the *Veech group* of X and is denoted by $SL(X)$. In particular, affinely equivalent surfaces have a conjugated Veech group. As well as introducing the notion (although not the name) W.A. Veech showed in [23] that they are discrete subgroups of $\mathrm{SL}_2(\mathbb{R})$. Hence, we can classify elements of the Veech group into three types: elliptic ($|\mathrm{tr}(M)| < 2$), parabolic ($|\mathrm{tr}(M)| = 2$), and hyperbolic ($|\mathrm{tr}(M)| > 2$). Any element of the Veech group induces a diffeomorphism of the surface. Such diffeomorphisms are called *affine diffeomorphisms.*

Trace field. — The *trace field* of a group $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ is the subfield of $\mathbb{R}$ generated over $\mathbb{Q}$ by $\{\mathrm{tr}(M), M \in \Gamma\}$. One defines the trace field of a translation surface to be the trace field of its Veech group.

Let X be a genus g translation surface. We have the following theorems:

THEOREM 2.2 (see [17]). — *The trace field of X has degree at most g over $\mathbb{Q}$.*

Assume the Veech group of X contains a hyperbolic element M. Then the trace field is exactly $\mathbb{Q}[tr(M)]$.

It is a classical result (see, for instance, [22]) that after a normalization, there exists an atlas such that every parabolic direction has its slope in the trace field, and every connection point has coordinates in the trace field. Specifically in the quadratic case, we have the following result:

THEOREM 2.3 ([18], Theorem 5.1, see also [3]). — *If the trace field is quadratic over $\mathbb{Q}$, then every direction whose slope lies in the trace field is parabolic.*

3. Hyperbolic directions for the double $(2n + 1)$-gon

I. Bouw and M. Möller in [4] gave a large class of Veech surfaces. W.P. Hooper gave a geometric interpretation of these surfaces in [14] and proved in particular that the double $(2n+1)$-gon is affinely equivalent to a staircase polygonal model. See also [6], [9], and [20]. See Figure 3.1 for the double heptagon's staircase model. We will use this model to construct the continued fractions algorithm at the heart of this paper, which is a direct generalization of that described in [8] in the setting of the double pentagon. For more results on the double pentagon, see also [7].

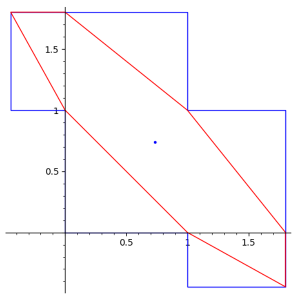

FIGURE 3.1. The staircase model for the double heptagon (in red we show one of the two heptagons).

The staircase model can be constructed as follows : Let each $R_i, i = 1, \dots, 2n-1$ be the rectangle of side $\sin(\frac{i\pi}{2n+1})$ and $\sin(\frac{(i+1)\pi}{2n+1})$. Glue R_i and R_{i+1} such that edges of the same size are glued together, each side being glued to the opposite side of the other rectangle as shown in Figure 3.2. Parallel edges of R_1 (or R_{2n-1}) that are not glued to an edge of another rectangle are glued together.

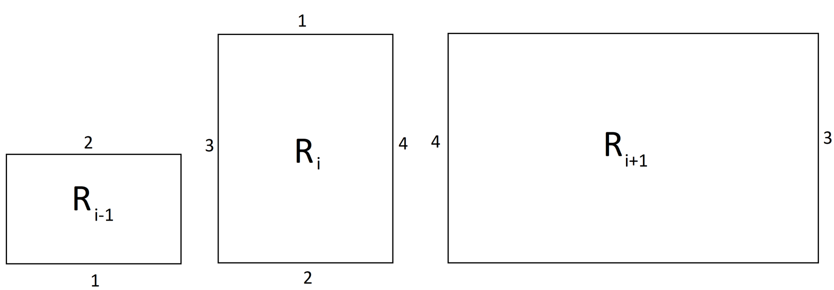

FIGURE 3.2. How to glue the rectangles R_i. Each edge of R_i is glued to the one with the same number in R_{i-1} or R_{i+1}.

It is then an easy calculation to establish the following lemma, which, in fact, is a particular case of Lemma 6.6 from [6] (see also [23]).

LEMMA 3.1. — *Let $n \geq 2$ be an integer. Then in the staircase model for the double $(2n+1)$-gon translation surface, there is a horizontal (or vertical)* *decomposition into cylinders such that all cylinders have modulus equal to* $a_n = 2\cos(\frac{\pi}{2n+1})$.

In fact, for computational reasons, it will be more convenient to rescale the staircase by a factor $\frac{1}{sin(\frac{n\pi}{2n+1})}$, so that each side can be expressed in the trace field, and the longer side has length 1.

Let us now look at the short diagonals of the staircase. We get $2n-1$ short diagonal vectors denoted by $D_i, i \in [\![1, 2n-1]\!]$. We set D_0 to be the shortest horizontal vector and D_{2n} the shortest vertical vector. We rescale such that D_0 and D_{2n} are length 1 vectors. We drew the diagonals in a graph as shown in Figure 3.3 for the double heptagon ($n = 3$). All the D_i's have a Euclidean norm bigger than 1 (except D_0 and D_{2n} with norm equal to 1).

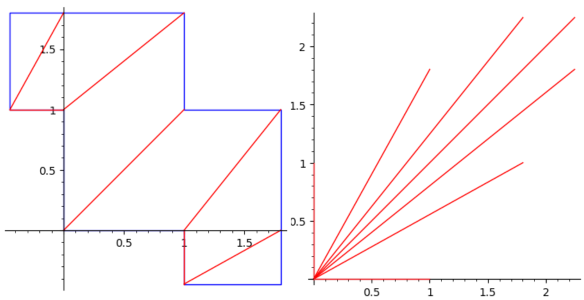

FIGURE 3.3. The diagonals of the double heptagon staircase divide the positive cone into six subcones. The diagonals are rescaled so that D_0 and D_{2n} are length 1 vectors. We have $D_0 = (1, 0)$, $D_1 = (a_3, 1)$, $D_2 = (a_3^2 - 1, a_3)$, $D_3 = (a_3^2 - 1, a_3^2 - 1)$, and the other diagonals are symmetrical about the first bisector.

Let M_i, $i \in [\![0, 2n-1]\!]$ be the matrix that maps $D_0 = (1, 0)$ to D_i and $D_{2n} = (0, 1)$ to D_{i+1}. Let Σ denote the first quadrant, and Σ_i its image under M_i (we include D_i in M_i). The matrix M_i is in the Veech group of the staircase and is associated to an affine homeomorphism of the staircase surface, which we still denote by M_i. This homeomorphism sends parabolic (or hyperbolic) directions[2] to parabolic (or hyperbolic) directions that are in the i^{th} cone. In fact, these matrices M_i already appear in [21]. Iterating this process, we obtain a way to construct new parabolic (or hyperbolic) directions once we have found one. Conversely, we have a continued fractions algorithm given by the following definition.

2. Here and throughout, by direction we mean an element of the projective line $\mathbb{P}(\mathbb{R}^2)$.

Definition 3.2 (continued fractions algorithm for the staircase model). — Given a direction in the first quadrant as the entry, apply the following procedure:

1) If the direction lies in the i^{th} cone, apply M_i^{-1}.
2) If the direction is neither horizontal nor vertical, go back to step 1.

The following theorem is due to D. Davis and S. Lelièvre. It is stated in [8] in the case of the double pentagon, but the same arguments can be directly extended to the double $(2n+1)$-gon.

Theorem 3.3 ([8]). — *A direction on the double* $(2n+1)$*-gon is parabolic if and only if the continued fractions algorithm terminates at the horizontal direction.*

This theorem gives the first possibility for this algorithm to end. The other possibility would be an eventually periodic ending, i.e., if we apply the algorithm a certain number of times, the direction we get is a direction that we already got in a previous step. Here, we characterize these directions in the trace field and we prove Theorem 1.1, which can be stated more formally in the following way:

Theorem 3.4. — *The continued fractions algorithm is eventually periodic for a direction* θ *(which is neither horizontal nor vertical) in the trace field if and only if* θ *is the image by a matrix* $M_{i_k}\dots M_{i_1}$ *of an eigendirection for a hyperbolic matrix of the form* $M_{j_1}\dots M_{j_l}$*. In particular, every eventually periodic direction for the continued fractions algorithm is an eigendirection for a hyperbolic matrix of the Veech group.*

Proof. — If θ is eventually periodic for the algorithm, let k denote the length of the preperiod of θ. Then, we have matrices $M_{i_1},\dots,M_{i_k}$, such that $\theta' = (M_{i_k}\dots M_{i_1})^{-1}(\theta)$ is periodic for the algorithm. That is, there exist $M_{j_1},\dots,M_{j_l}$ such that $M_{j_1}\dots M_{j_l}(\theta') = \theta'$. Then $M = M_{j_1}\dots M_{j_l}$ is, indeed, a hyperbolic matrix since all M_js dilate lengths in the first quadrant, which means that the eigenvalue of $M_{j_1}\dots M_{j_l}$ for the direction θ' has to be strictly bigger than 1. Moreover, M belongs to the Veech group, being a product of elements of the Veech group.

Conversely, let us suppose that there are $i_1,\dots,i_k,j_1,\dots,j_l$ such that $M_{j_1}\dots M_{j_l}(\theta') = \theta'$, where $M = M_{j_1}\dots M_{j_l}$ is hyperbolic and $\theta = M_{i_k}\dots M_{i_1}(\theta')$. First, it is clear that θ' belongs to the first quadrant by the Perron–Frobenius theorem since all the matrices M_i have positive entries, and that the only sequences $j_1,\dots,j_l$ such that $M = M_{j_1}\dots M_{j_l}$ have possible zero entries are if $j_1 = \dots = j_l = 0$ or $j_1 = \dots = j_l = 2n$, which gives a matrix M that is parabolic and not hyperbolic. Thus, θ belongs to the first quadrant as well because the M_i's are contractions of the first quadrant. Moreover, at every step q, $M_{i_q}\dots M_{i_1}(\theta')$ belongs to the first quadrant. By construction of the algorithm, it follows that applying the algorithm to the direction θ leads to θ'

after k steps. By the same argument, since $M_{j_1} \dots M_{j_l}(\theta') = \theta'$ and θ' belongs to the first quadrant, we conclude that the sequence $j_l, \dots, j_1$ is exactly the sequence of indices we would have got if we had applied the algorithm to θ', and that θ' is a periodic direction for the algorithm. Hence, θ is an eventually periodic direction for the algorithm. □

Remark 3.5. — A point worth noting is that the sequence of sectors along the algorithm allows us to construct the matrix M, which stabilizes the original direction. This will allow us, for the double heptagon, to find a separatrix whose direction is eventually periodic for the algorithm and, hence, is not parabolic, which means that the separatrix does not extend to a saddle connection.

Example 3.6. — For the continued fractions algorithm on the double heptagon:

- The direction of slope $a_3^2 - 1$ is 2-periodic and fixed by the hyperbolic matrix $M_5 M_0$.
- The direction of slope $\frac{39}{7}a_3^2 + \frac{30}{7}a_3 - \frac{19}{7}$ is 28-periodic and fixed by the hyperbolic matrix $M_5^{12} M_4^2 M_0^{12} M_2 M_0$.

4. Connection points

In this section, we finally show that central points of the double heptagon are not connection points. We first give some motivation to their study.

Connection points have been studied in [15] by P. Hubert and T. Schmidt, who gave a construction of translation surfaces with infinitely generated Veech groups as branched covers over nonperiodic connection points. C. McMullen proved the existence of these points in [19] in the case of a quadratic trace field and implicitly showed that the connection points are exactly the points with coordinates in the trace field. However, in a higher degree there is no such result, neither concerning connection points nor about infinitely generated Veech groups. One of the easiest nonquadratic surfaces is the double heptagon, whose trace field is of degree 3 over $\mathbb{Q}$. P. Arnoux and T. Schmidt implicitly showed (see [2]) that for the double heptagon surface there are points with coordinates in the trace field that are not connection points. Still, it was not known whether or not central points of the double heptagon were connection points. Here, we provide a negative answer to this question.

By definition, for proving that a point is not a connection point, it suffices to find a separatrix passing through it, which cannot be extended to a saddle connection, for instance because the separatrix lies in a hyperbolic direction. We managed to find such a separatrix for a central point, which is drawn in Figure 4.1. Of course, both central points play a symmetric role, so it suffices to consider either one of them.