Bulletin
de la SOCIÉTÉ MATHÉMATIQUE DE FRANCE

THE GENERALIZED INJECTIVITY CONJECTURE

Sarah Dijols

Tome 150
Fascicule 2

2022

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

pages 251-345

Le *Bulletin de la Société Mathématique de France* est un périodique trimestriel
de la Société Mathématique de France.

Fascicule 2, tome 150, juin 2022

Diffusion

Maison de la SMF
Case 916 - Luminy
13288 Marseille Cedex 9
France
commandes@smf.emath.fr

AMS
P.O. Box 6248
Providence RI 02940
USA
www.ams.org

Tarifs

Vente au numéro : 43 € ($ 64)
Abonnement électronique : 135 € ($ 202),
avec supplément papier : Europe 179 €, hors Europe 197 € ($ 296)
Des conditions spéciales sont accordées aux membres de la SMF.

Secrétariat : Bulletin de la SMF

Bulletin de la Société Mathématique de France
Société Mathématique de France
Institut Henri Poincaré, 11, rue Pierre et Marie Curie
75231 Paris Cedex 05, France
Tél : (33) 1 44 27 67 99 • Fax : (33) 1 40 46 90 96
bulletin@smf.emath.fr • smf.emath.fr

ISSN 0037-9484 (print) 2102-622X (electronic)

Directeur de la publication : Fabien DURAND

Bull. Soc. Math. France
150 (2), 2022, p. 251-345

THE GENERALIZED INJECTIVITY CONJECTURE

by Sarah Dijols

Abstract. — We prove a conjecture of Casselman and Shahidi stating that the unique irreducible generic subquotient of a standard module is necessarily a subrepresentation for a large class of connected quasi-split reductive groups, in particular for those that have a root system of classical type (or product of such groups). To do so, we prove and use the existence of strategic embeddings for irreducible generic discrete series representations, extending some results of Moeglin.

Résumé (*La conjecture d'injectivité généralisée*). — Nous prouvons la conjecture de Casselman-Shahidi, qui affirme que l'unique sous-quotient générique d'un module standard est nécéssairement une sous-représentation, pour une large classe de groupes réductifs, quasi-déployés et connexes, en particulier ceux qui ont un système de racines de type classique (ou produit de tels groupes). Pour se faire, nous prouvons l'existence de certains plongements particuliers de représentations séries discrètes, généralisant ainsi des résultats de Moeglin.

Texte reçu le 24 juin 2020, modifié le 4 août 2021, accepté le 11 janvier 2022.

Sarah Dijols, 312 Jingzhai, Tsinghua University, Qinghuqyuan street, Haidian District, Beijing, China • *E-mail :* sarah.dijols@hotmail.fr

Mathematical subject classification (2010). — 11F70, 22E50.

Key words and phrases. — Representations of p-adic groups, Whittaker models, generic subquotients, standard module.

This work is part of the author's PhD thesis under the supervision of Volker Heiermann, at Aix-Marseille University. The author has benefited from a grant of Agence Nationale de la Recherche with reference ANR-13-BS01-0012 FERPLAY..

doi:10.24033/bsmf.2848

1. Introduction

1.1. Let G be a quasi-split connected reductive group over a non-Archimedean local field F of characteristic zero. We assume that we are given a standard parabolic subgroup P with Levi decomposition $P = MU$, as well as an irreducible, tempered, generic representation τ of M. Now let ν be an element in the dual of the real Lie algebra of the split component of M; we take it in the positive Weyl chamber. The induced representation $I_P^G(\tau, \nu) := I_P^G(\tau_\nu)$, called the standard module, has a unique irreducible quotient, $J(\tau_\nu)$, often named the Langlands quotient. Since the representation τ is generic (for a non-degenerate character of U, see Section 2), i.e. has a Whittaker model, the standard module $I_P^G(\tau_\nu)$ is also generic. Further, by a result of Rodier [29] any generic induced module has a unique irreducible generic subquotient.

In their paper, Casselman and Shahidi [7] conjectured that:

(A) $J(\tau_\nu)$ is generic if and only if $I_P^G(\tau_\nu)$ is irreducible.
(B) The unique irreducible generic subquotient of $I_P^G(\tau_\nu)$ is a subrepresentation.

These questions were originally formulated for real groups by Vogan [38]. Conjecture (B), was resolved in [7] provided the inducing data is cuspidal. Conjecture (A), known as the standard module conjecture, was first proven for classical groups by Muić in [26] and was settled for quasi-split p-adic groups in [18] assuming the tempered L function conjecture proven a few years later in [19].

The second conjecture, known as the generalized injectivity conjecture was proved for classical groups $\mathrm{SO}(2n+1), Sp(2n)$, and $\mathrm{SO}(2n)$ for P a maximal parabolic subgroup, by Hanzer in [13].

In the present work, we prove the generalized injectivity conjecture (Conjecture (B)) for a large class of quasi-split connected reductive groups provided that the irreducible components of a certain root system (denoted Σ_σ) are of type A, B, C or D (see Theorem 1.1 below for a precise statement). Following the terminology of Borel–Wallach [4.10 in [3]], for a standard parabolic subgroup P, τ a tempered representation and $\eta \in (a_M^*)^+$, a positive Weyl chamber, (P, τ, η) is referred as Langlands data, and η is the Langlands parameter, see the Definition 2.8 herein.

We will study the unique irreducible generic subquotient of a standard module $I_P^G(\tau_\eta)$ and *first* make the following reductions:

- τ is a discrete series representation of the standard Levi subgroup M
- P is a maximal parabolic subgroup.
  Then, η is written $s\tilde{\alpha}$, see Section 1.4 for a definition of the latter.

Then, our approach has two layers. First, we realize the generic discrete series τ as a subrepresentation of an induced module $I_{P_1\cap M}^M(\sigma_\nu)$ for a unitary generic cuspidal representation of M_1 (using Proposition 2.5 of [19]), and the parameter

ν is dominant (i.e. in some positive closed Weyl chamber) in a sense later made precise; Using induction in stages, we can therefore embed the standard module $I_P^G(\tau_{s\tilde{\alpha}})$ in $I_{P_1}^G(\sigma_{\nu+s\tilde{\alpha}})$.

Let us denote $\nu + s\tilde{\alpha} := \lambda$. The unique generic subquotient of the standard module is also the unique generic subquotient in $I_{P_1}^G(\sigma_\lambda)$. By a result of Heiermann–Opdam [Proposition 2.5 of [19]], this generic subquotient appears as a subrepresentation of yet another induced representation $I_{P'}^G(\sigma'_{\lambda'})$ characterized by a parameter λ' in the closure of some positive Weyl chamber.

In an ideal scenario, λ and λ' are dominant with respect to P_1 (resp. P'), i.e. λ and λ' are in the closed positive Weyl chamber, and we may then build a bijective operator between those two induced representations using the dominance property of the Langlands parameters.

In case the parameter λ is not in the closure of the positive Weyl chamber, two alternatives procedures are considered: first, another strategic embedding of the irreducible generic subquotient in the representation induced from $\sigma''_{\lambda''}$ (relying on extended Moeglin's Lemmas) when the parameter λ'' (which depends on the form of λ) has a very specific aspect (this is Proposition 6.14); or (resp. and) showing the intertwining operator between $I_{P'}^G(\sigma'_{\lambda'})$ (or $I_{P_1}^G(\sigma''_{\lambda''})$) and $I_{P_1}^G(\sigma_\lambda)$ has a non-generic kernel.

1.2. In order to study a larger framework than the one of classical groups studied in [13], we will use the notion of *residual points* of the μ function (the μ function is the main ingredient of the Plancherel density for p-adic groups (see the Definition 2.4 and Section 2.2).

Indeed, as briefly suggested in the previous point, the triple (P_1, σ, λ), introduced above, plays a pivotal role in all the arguments developed thereafter, and of particular importance, the parameter λ is related to the μ function in the following ways:

- When σ_λ is a residual point for the μ function (abusively one says that λ is a residual point once the context is clear), the unique irreducible generic subquotient in the module induced from σ_λ is discrete series (a result of Heiermann in [15], see Proposition 2.6).
- Once the cuspidal representation σ is fixed, we attach to it the set Σ_σ, a root system in a subspace of $a^*_{M_1}$ defined using the μ function. More precisely, let α be a root in the set of reduced roots of A_{M_1} in Lie(G) and $(M_1)_\alpha$ be the centraliser of $(A_{M_1})_\alpha$ (the identity component of the kernel of α in A_{M_1}). We will consider the set

$$\Sigma_\sigma = \{\alpha \in \Sigma_{\text{red}}(A_{M_1}) | \mu^{(M_1)_\alpha}(\sigma) = 0\}$$

  It is a subset of $a^*_{M_1}$ which is a root system in a subspace of $a^*_{M_1}$(cf. [35] 3.5) and we suppose the irreducible components of Σ_σ are of type A, B, C or D. Let us denote W_σ the Weyl group of Σ_σ.

This is where stands the particularity of our method, to deal with all possible standard modules, we needed an explicit description of this parameter λ lying in $a^*_{M_1}$. Thanks to Opdam's work in the context of affine Hecke algebras and Heiermann's one in the context of p-adic reductive groups such descriptive approach is made possible. Indeed, we have a bijective correspondence between the following sets explained in Section 4: {dominant residual point} $\leftrightarrow$ Weighted Dynkin diagram(s)}

The notion of Weighted Dynkin diagram is established and recalled in the Appendix A.1. We use this correspondence to express the coordinates of the dominant residual point and name this expression of the residual point a *residual segment* generalizing the classical notion of segments (of Bernstein–Zelevinsky). We associate to such a residual segment *set(s) of jumps* (a notion connected to that of Jordan block elements in the classical groups setting of Moeglin–Tadić in [23]).

Further, the μ function is intrinsically related to the *intertwining operators* mentioned in the previous subsection. A key aspect of this work is an appropriate use of (standard) intertwining operators, more precisely the use of intertwining operators with a non-generic kernel. Using the functoriality of induction, it is always possible to reduce the study of intertwining operators to *rank 1* intertwining operators (i.e. consider the well-understood intertwining operator $J_{s_{\alpha_i}P_1|P_1}$ between $I^{M_1}_{P_1\cap(M_1)_{\alpha_i}}(\sigma_\lambda)$ and $I^{M_1}_{\overline{P_1}\cap(M_1)_{\alpha_i}}(\sigma_\lambda)$); and in particular if σ is irreducible cuspidal (see Theorem 2.1). At the level of rank 1 intertwining operator (where $I^{M_1}_{P_1\cap(M_1)_{\alpha_i}}(\sigma_\lambda)$ is the direct sum of two non-isomorphic representations, see Theorem 2.1), determining the non-genericity of the kernel of the map $J_{s_{\alpha_i}P_1|P_1}$ reduces to a simple condition on the relevant coordinates (i.e. the coordinates determined by α_i) of $\lambda \in a^*_{M_1}$.

1.3. Having defined the root system Σ_σ let us present the main result of this paper:

Theorem 1.1 (Generalized injectivity conjecture for a quasi-split group). — *Let G be a quasi-split, connected group defined over a p-adic field F (of characteristic zero) such that its root system is of type A, B, C or D (or the product of these). Let π_0 be the unique irreducible generic subquotient of the standard module $I^G_P(\tau_\nu)$; then π_0 embeds as* a subrepresentation *in the standard module $I^G_P(\tau_\nu)$.*

Theorem 1.2 (Generalized injectivity conjecture for quasi-split group). — *Let G be a quasi-split, connected group defined over a p-adic field F (of characteristic zero). Let π_0 be the unique, irreducible generic subquotient of the standard module $I^G_P(\tau_\nu)$ and let σ be an irreducible, generic, cuspidal representation of*

M_1 such that a twist by an unramified real character of σ is in the cuspidal support of π_0.

Suppose that all the irreducible components of Σ_σ are of type A, B, C or D; then, under certain conditions on the Weyl group of Σ_σ (which is explained in Section 6.1, in particular Corollary 6.6), π_0 embeds as a subrepresentation *in the standard module $I_P^G(\tau_\nu)$.*

Theorem 1.1 results from 1.2. Theorem 1.2 is true when the root system of the group G contains components of type E, F provided that Σ_σ is irreducible and of type A. We do not know if an analogue of Corollary 6.6 holds for groups whose root systems are of type E or F. Further, in the exceptional groups of type E or F, many cases where the cuspidal support of π_0 is (P_0, σ) (generalized principal series) cannot be dealt with using the methods proposed in this work; see Section 9 for details.

1.4. Let us briefly comment on the organisation of this manuscript, therefore giving a general overview of our results and the scheme of proof.

In Section 3, we formulate the problem in an as broad as possible context (any quasi-split reductive p-adic group G) and prove a few results on intertwining operators.

As M. Hanzer in [13], we distinguish two cases: the case of a generic discrete series subquotient and the case of a non-discrete series generic subquotient. As stated in 1.2, the case of a discrete series subquotient corresponds to σ_λ (in the cuspidal support of the generic discrete series) being a residual point.

As just stated in 1.2, our approach uses the bijection between Weyl group orbits of residual points and weighted Dynkin diagrams as studied in [27] and explained in the Appendix A.

Through this approach, we can make explicit the Langlands parameters of subquotients of the representations $I_{P_1}^G(\sigma_\lambda)$ induced from the generic cuspidal support σ_λ and classify them using the order on parameters in $a_{M_1}^*$ as given in Chapter XI, Lemma 2.13 in [3]. In particular, the minimal element for this order (in a sense that is later made precise) characterizes the unique irreducible generic non-discrete series subquotient; see Theorem 5.5.

Although requiring us to get acquainted with the notions of residual points, and then residual segments, our methods have two advantages.

The first is proving the generalized injectivity conjecture for a large class of quasi-split reductive groups (provided a certain construction of the standard Levi subgroup M_1 and the irreducible components of Σ_σ to be of type A, B, C or D; we have verified those conditions when the root system of the quasi-split (hence reductive) group is of type A, B, C or D), and recovering the results of Hanzer through alternative proofs. In particular, a key ingredient (which was not used by Hanzer in [13]) in our method is an embedding result of Heiermann–Opdam (Proposition 2.7). The second is a self-contained and uniform (in the sense that cases of root systems of type B, C and D are all treated in the same proofs) treatment.

Although based on the ideas of Hanzer in [13], our approach includes a much larger class of quasi-split groups and some cases of exceptional groups.

We separate this work into two different problems. The first problem is to determine the conditions on $\lambda \in a_{M_1}^*$ so that the unique generic subquotient of $I_{P_1}^G(\sigma_\lambda)$ with σ irreducible unitary generic cuspidal representation of a standard Levi M_1 is a subrepresentation. The results on this problem are presented in Theorem 6.3.

The second problem is to show that any standard module can be embedded in a module induced from cuspidal generic data, with $\lambda \in a_{M_1}^*$ satisfying one of the conditions mentioned in Theorem 6.3. This is done in the Section 7 and the following.

Regarding the first problem: in the Section 6.3, we present an embedding result for the unique irreducible generic discrete series subquotient of the generic standard module (see Proposition 6.14) relying on two extended Moeglin's lemmas (see Lemmas 6.12 and 6.13) and the result of Heiermann–Opdam (see Proposition 2.7). This embedding and the use of standard intertwining operators with a non-generic kernel allow us to prove the Theorem 6.3.

Once achieved the Theorem 6.3; it is rather straightforward to prove the generalized injectivity conjecture for the discrete series generic subquotient, first when P is a maximal parabolic subgroup and secondly for *any parabolic subgroup* in Section 7.2.

In Section 7.3, we continue with the case of a generic non-discrete series subquotient and further conclude with the case of the standard module induced from a tempered representation τ in Corollary 7.11 and Corollary 8.3.

The proof of Theorem 1.2 is done in several steps. First, we prove it for the case of an irreducible generic discrete series subquotient assuming τ discrete series and Σ_σ irreducible in Proposition 7.3.

We use this latter result for the case of a non-square integrable irreducible generic subquotient in Proposition 7.9; and also for the case of standard modules induced from non-maximal standard parabolic (Theorems 7.8 and 7.10). Then, the case of τ tempered follows (Corollary 7.11). The case of Σ_σ reducible is done in Section 8 and relies on the Appendix B.

The reader familiar with the work of Bernstein–Zelevinsky on GL_n (see [30] or [40]) may want to have a look at the author's PhD thesis where we treat independently the case of Σ_σ of type A to get a quicker overview on some tools used in this work.

From here, we use the following notations:

Notation. —
- *Standard module induced from a maximal parabolic subgroup:* Let $\Theta = \Delta - \{\alpha\}$ for α in Δ and let $P = P_\Theta$ be a maximal parabolic subgroup of G. We denote ρ_P the half sum of positive roots

in U, and for α the unique simple root for G, which is not a root for M,

$$\tilde{\alpha} = \frac{\rho_P}{(\rho_P, \alpha)} \quad \text{where } (\rho_P, \alpha) = \frac{2\langle \rho_P, \alpha \rangle}{\langle \alpha, \alpha \rangle}.$$

(Rather than $\tilde{\alpha}$, in the split case, we could also take the fundamental weight corresponding to α). Since ν is in a_M^* (of dimension rank(G) − rank$(M) = 1$ since M is maximal) and should satisfy $\langle \nu, \check{\beta} \rangle > 0$, for all $\beta \in \Delta - \Theta = \{\alpha\}$, the standard module in this case is $I_P^G(\tau_{s\tilde{\alpha}})$, where $s \in \mathbb{R}$ such that $s > 0$, and τ is an irreducible tempered representation of M.

- For the sake of readability, we sometimes denote $I_{P_1}^G(\sigma(\lambda)) := I_{P_1}^G(\sigma_\lambda)$ when the parameter λ is expressed in terms of residual segments.
- Let σ be an irreducible cuspidal representation of a Levi subgroup $M_1 \subset M$ in a standard parabolic subgroup P_1, and let λ be in $(a_{M_1}^*)$; we denote $Z^M(P_1, \sigma, \lambda)$ the unique irreducible generic discrete series (or essentially square-integrable) in the standard module $I_{P_1 \cap M}^M(\sigma_\lambda)$.

  We will omit the index when the representation is a representation of G: $Z(P_1, \sigma, \lambda)$; often λ will be written explicitly with residual segments to emphasize the dependency on specific sequences of exponents.

2. Preliminaries

2.1. Basic objects. — Throughout this paper we will let F be a non-Archimedean local field of characteristic 0. We will denote by G the group of F-rational points of a quasi-split connected reductive group defined over F. We fix a minimal parabolic subgroup P_0 (which is a Borel B since G is quasi-split) with Levi decomposition $P_0 = M_0 U_0$ and A_0 a maximal split torus (over F) of M_0; P is said to be standard if it contains P_0. More generally, if P rather contains A_0, it is said to be semi-standard. Then P contains a unique Levi subgroup M containing A_0, and M is said to be semi-standard. For a semi-standard Levi subgroup M, we denote $\mathscr{P}(M)$ the set of parabolic subgroups P with Levi factor M.

We denote by A_M the maximal split torus in the center of M, $W = W^G$ the Weyl group of G defined with respect to A_0 (i.e. $N_G(A_0)/Z_G(A_0)$). The choice of P_0 determines an order in W, and we denote by w_0^G the longest element in W.

If Σ denote the set of roots of G with respect to A_0, the choice of P_0 also determines the set of positive roots (or negative roots, simple roots) which we denote by Σ^+ (or Σ^-, Δ).

To a subset $\Theta \subset \Delta$ we associate a standard parabolic subgroup $P_\Theta = P$ with Levi decomposition MU and denote A_M the split component of M. We will write a_M^* for the dual of the real Lie-algebra a_M of A_M, $(a_M)_{\mathbb{C}}^*$ for its complexification and a_M^{*+} for the positive Weyl chamber in a_M^* defined with respect to P. Further, $\Sigma(A_M)$ denotes the set of roots of A_M in Lie(G). It is a subset of a_M^*. For any root $\alpha \in \Sigma(A_M)$, we can associate a coroot $\check{\alpha} \in a_M$. For $P \in \mathcal{P}(M)$, we denote $\Sigma(P)$ the subset of positive roots of A_M relative to P.

Let Rat(M) be the group of F-rational characters of M; we have:

$$a_M^* = \mathrm{Rat}(M) \otimes_{\mathbb{Z}} \mathbb{R} \text{ and } (a_M)^*_{\mathbb{C}} = a_M^* \otimes_{\mathbb{R}} \mathbb{C}.$$

For $\chi \otimes r \in a_M^*$, $r \in \mathbb{R}$, and λ in a_M, the pairing $a_M \times a_M^* \to \mathbb{R}$ is given by: $\langle \lambda, \chi \otimes r \rangle = \lambda(\chi) \cdot r$

Following [39] we define a map

$$H_M : M \to a_M = \mathrm{Hom}(\mathrm{Rat}(M), \mathbb{R})$$

such that

$$|\chi(m)|_F = q^{-\langle \chi, H_M(m) \rangle},$$

for every F-rational character χ in a_M^* of M, q being the cardinality of the residue field of F. Then H_P is the extension of this homomorphism to P, extended trivially along U.

We denote by $X(M)$ the group of unramified characters of M.

Let us assume that (σ, V) is an admissible complex representation of M. We adopt the convention that the isomorphism class of (σ, V) is denoted by σ. If χ_ν is in $X(G)$, with $\nu \in a_{G,\mathbf{C}}^*$, then we write $(\sigma_\nu, V_{\chi_\nu})$ for the representation $\sigma \otimes \chi_\nu$ on the space V.

Let (σ, V) be an admissible representation of finite length of M, a Levi subgroup containing M_0 a minimal Levi subgroup, centraliser of the maximal split torus A_0. Let P and P' be in $\mathcal{P}(M)$. Consider the intertwining integral:

$$(J_{P'|P}(\sigma_\nu)f)(g) = \int_{U \cap U' \backslash U'} f(u'g)du' \quad f \in I_P^G(\sigma_\nu),$$

where U and U' denote the unipotent radical of P and P', respectively.

For ν in $X(M)$ with $\mathrm{Re}(\langle \nu, \check{\alpha} \rangle) > 0$, for all α in $\Sigma(P) \cap \Sigma(P')$, the defining integral of $J_{P'|P}(\sigma_\nu)$ converges absolutely. Moreover, $J_{P'|P}$ defined in this way on some open subset of $\mathcal{O} = \{\sigma_\nu | \nu \in X(M)\}$ becomes a rational function on $\mathcal{O}$ ([39] Theorem IV 1.1). Outside its poles, this defines an element of

$$\mathrm{Hom}_G(I_P^G(V_\chi), I_{P'}^G(V_\chi)).$$

Moreover, for any χ in $X(M)$, there exists an element v in $I_P^G(V_\chi)$ such that $J_{P'|P}(\sigma_\chi)v$ is not zero ([39], IV.1 (10))

In particular, for all ν in an open subset of a_M^*, and $\overline{P}$ the opposite parabolic subgroup to P, we have an intertwining operator

$$J_{\overline{P}|P}(\sigma_\nu) : I_P^G(\sigma_\nu) \to I_{\overline{P}}^G(\sigma_\nu)$$