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THE NAKAYAMA FUNCTOR AND ITS COMPLETION FOR GORENSTEIN ALGEBRAS

BY SRIKANTH B. IYENGAR & HENNING KRAUSE

To Bill Crawley-Boevey on his 60th birthday.

ABSTRACT. — Duality properties are studied for a Gorenstein algebra that is finite and projective over its center. Using the homotopy category of injective modules, it is proved that there is a local duality theorem for the subcategory of acyclic complexes of such an algebra, akin to the local duality theorems of Grothendieck and Serre in the context of commutative algebra and algebraic geometry. A key ingredient is the Nakayama functor on the bounded derived category of a Gorenstein algebra and its extension to the full homotopy category of injective modules.

RÉSUMÉ (*Le foncteur de Nakayama et sa complétion pour les algèbres de Gorenstein*). — Des propriétés de dualité sont étudiées pour une algèbre de Gorenstein finie et projective sur son centre. En utilisant la catégorie homotopique des modules injectifs, il est démontré qu'il existe un théorème de dualité locale pour la sous-catégorie des objets acycliques d'une telle algèbre, semblable aux théorèmes de dualité locale de Grothendieck et Serre dans le cadre de l'algèbre commutative et de la géométrie algébrique. Un ingrédient clé est le foncteur de Nakayama sur la catégorie dérivée bornée d'une algèbre de Gorenstein, et son extension à toute la catégorie homotopique des modules injectifs.

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1. Introduction

This work is a contribution to the representation theory of Gorenstein algebras, both commutative and noncommutative, with a focus on duality phenomena. The notion of a Gorenstein variety was introduced by Grothendieck [26, 25, 29, 30] and grew out of his reinterpretation and extension of Serre duality [43] for projective varieties. A local version of his duality is that over a Cohen–Macaulay local algebra R of dimension d, with maximal ideal $\mathfrak{m}$, and for complexes F, G with F perfect, there are natural isomorphisms

$$\operatorname{Hom}_R(\operatorname{Ext}^i_R(F,G), I(\mathfrak{m})) \cong \operatorname{Ext}^{d-i}_R(G, R\varGamma_{\mathfrak{m}}(\omega_R \otimes^{\mathrm{L}}_R F)),$$

where ω_R is a dualizing module, and $I(\mathfrak{m})$ is the injective envelope of $R/\mathfrak{m}$. The functor $R\varGamma_{\mathfrak{m}}$ represents local cohomology at $\mathfrak{m}$. Serre duality concerns the case where R is the local ring at the vertex of the affine cone of a projective variety. The ring R (equivalently, the variety it represents) is said to be Gorenstein if, in addition, the R-module ω_R is projective. Serre observed that this property is characterized by R having a finite self-injective dimension. This result appears in the work of Bass [4], who gave numerous other characterizations of Gorenstein rings.

Iwanaga [31] launched the study of Noetherian rings, not necessarily commutative, having finite self-injective dimension on both sides. Now known as Iwanaga–Gorenstein rings, these form an integral part of the representation theory of algebras. In that domain, the principal objects of interest are maximal Cohen–Macaulay modules and the associated stable category. Auslander [1] and Buchweitz [13] have proved duality theorems for the stable category of a Gorenstein algebra with *isolated* singularities. The driving force behind our work was to understand what duality phenomena can be observed for general Gorenstein algebras. Theorem 1.2 below is what we found, following Grothendieck's footsteps.

We set the stage to present that result and begin with a crucial definition.

Definition 1.1. — Let R be a commutative Noetherian ring. An R-algebra A is called *Gorenstein* if

(1) the R-module A is finitely generated and projective, and
(2) for each $\mathfrak{p}$ in $\operatorname{Spec} R$ with $A_{\mathfrak{p}} \neq 0$ the ring $A_{\mathfrak{p}}$ has finite injective dimension as a module over itself, on the left and on the right.

A Gorenstein R-algebra A itself need not be Iwanaga–Gorenstein. Indeed, for A commutative and Gorenstein, the injective dimension of A is finite precisely when its Krull dimension is finite, and there exist rings locally of finite injective dimension but of infinite Krull dimension. There are precedents to the study of Gorenstein algebras, starting with [4] and more recently in the work of Goto and Nishida [24]. Our work differs from theirs in its focus on duality. We refer to [22] for a discussion of examples and natural constructions preserving the Gorenstein property.

Let A be a Gorenstein R-algebra and $\omega_{A/R} := \operatorname{Hom}_R(A,R)$ the dualizing *bimodule*. Unlike in the commutative case, $\omega_{A/R}$ does not need to be projective (neither on the left nor on the right), and the bimodule structure can be complicated. Nevertheless, it is a tilting object in $\mathbf{D}(\operatorname{Mod} A)$, the derived category of A-modules, inducing a triangle equivalence

$$\operatorname{RHom}_A(\omega_{A/R}, -)\colon \mathbf{D}(\operatorname{Mod} A) \xrightarrow{\sim} \mathbf{D}(\operatorname{Mod} A)\,;$$

see Section 4. The representation theory of a Gorenstein algebra A is governed by its maximal Cohen–Macaulay modules, namely, finitely generated A-modules M with $\operatorname{Ext}^i_A(M,A) = 0$ for $i \geq 1$. For our purposes, their infinitely generated counterparts are also important. Thus, we consider Gorenstein projective A-modules (abbreviated to G-projective), which are by definition A-modules occurring as syzygies in acyclic complexes of projective A-modules [13, 19]. The G-projective modules form a Frobenius exact category, and so the corresponding stable category, is triangulated. Its inclusion into the usual stable module category has a right adjoint, the Gorenstein approximation functor, $\operatorname{GP}(-)$. The functor

$$S := \operatorname{GP}(\omega_{A/R} \otimes_A -)\colon \underline{\operatorname{GProj}}A \longrightarrow \underline{\operatorname{GProj}}A$$

is an equivalence of triangulated categories and plays the role of a Serre functor on the subcategory of finitely generated G-projectives. This is spelled out in the result below. Here, the $\widehat{\operatorname{Ext}}^i_A(-,-)$ are the Tate cohomology modules, which compute morphisms in $\underline{\operatorname{GProj}}A$.

Theorem 1.2. — *Let A be a Gorenstein R-algebra and let M, N be G-projective A-modules with M finitely generated. For each $\mathfrak{p} \in \operatorname{Spec} R$, there is a natural isomorphism*

$$\operatorname{Hom}_R(\widehat{\operatorname{Ext}}^i_A(M,N), I(\mathfrak{p})) \cong \widehat{\operatorname{Ext}}^{d(\mathfrak{p})-i}_A(N, \varGamma_{\mathfrak{p}} S(M))\,,$$

where $d(\mathfrak{p}) = \dim(R_{\mathfrak{p}}) - 1$.

This is the duality theorem we seek; it is proved in Section 9. It is new even for commutative rings. The parallel to Grothendieck's duality theorem is clear.

In the following, we explain the strategy for proving this theorem and some essential ingredients. The functor $\varGamma_{\mathfrak{p}}$ is analogous to the local cohomology functor encountered above. It is constructed in Section 7 following the recipe in [7], using the natural R-action on $\underline{\operatorname{GProj}}A$. Even if N is finitely generated, $\varGamma_{\mathfrak{p}}(N)$ need not be, which is one reason we have to work with infinitely generated modules in the first place. If R is local with maximal ideal $\mathfrak{p}$, and A has isolated singularities, $\varGamma_{\mathfrak{p}}$ is the identity, and the duality statement above is precisely the one discovered by Auslander and Buchweitz.

For a Gorenstein algebra, the stable category of G-projective modules is equivalent to $\mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A)$, the homotopy category of acyclic complexes of injective A-modules. This connection is explained in Section 6 and builds on the results from [33, 35]. In fact, much of the work that goes into proving Theorem 1.2 deals with $\mathbf{K}(\operatorname{Inj} A)$, the full homotopy category of injective A-modules; see Section 2. A key ingredient in all this is the Nakayama functor on the category of A-modules:

$$\mathbf{N}\colon \operatorname{Mod} A \longrightarrow \operatorname{Mod} A \quad \text{where} \quad \mathbf{N}(M) = \operatorname{Hom}_A(\omega_{A/R}, M).$$

As noted above, its derived functor induces an equivalence on $\mathbf{D}(\operatorname{Mod} A)$. Following [35] we extend the Nakayama functor to all of $\mathbf{K}(\operatorname{Inj} A)$, which one may think of as a triangulated analogue of the ind-completion of $\mathbf{D}^{\mathrm{b}}(\operatorname{mod} A)$. This *completion* of the Nakayama functor is also an equivalence:

$$\widehat{\mathbf{N}}_{A/R}\colon \mathbf{K}(\operatorname{Inj} A) \xrightarrow{\sim} \mathbf{K}(\operatorname{Inj} A)\,.$$

This is proved in Section 5, where we establish also that it restricts to an equivalence on $\mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A)$. The induced equivalence on the stable category of G-projective modules is precisely the functor S in the statement of Theorem 1.2; see Section 6 where the singularity category of A, in the sense of Buchweitz [13] and Orlov [42] also appears. To make this identification, we need to extend results of Auslander and Buchweitz concerning G-approximations; this is dealt with in Appendix A.

Our debt to Grothendieck is evident. It ought to be clear by now that the work of Auslander and Buchweitz also provides much inspiration for this paper. Whatever new insight we bring is through the systematic use of the homotopy category of injective modules and methods from abstract homotopy theory, especially the Brown representability theorem. To that end we need the structure theory of injectives over finite R-algebras from Gabriel's thesis [20]. Gabriel also introduced the Nakayama functor in representation theory of Artin algebra in his exposition of Auslander–Reiten duality; it is the categorical analogue of the Nakayama automorphism that permutes the isomorphism classes of simple modules over a self-injective algebra [21]. Moreover, it was Gabriel who pointed out the parallel between derived equivalences induced by tilting modules and the duality of Grothendieck and Roos [34].

2. Homotopy category of injectives

In this section, we describe certain functors on homotopy categories attached to Noetherian rings. Our basic references for this material are [32, 35].

Throughout, A will be a ring that is Noetherian on both sides; that is to say, A is Noetherian as a left and as a right A-module. In what follows, A-modules will mean left A-modules, and A^{op}-modules are identified with right A-modules. We write $\operatorname{Mod} A$ for the (abelian) category of A-modules and $\operatorname{mod} A$ for its full subcategory consisting of finitely generated modules. Also, $\operatorname{Inj} A$ and $\operatorname{Proj} A$ are the full subcategories of $\operatorname{Mod} A$ consisting of injective and projective modules, respectively.

For any additive category $\mathcal{A} \subseteq \operatorname{Mod} A$, like the ones in the last paragraph, $\mathbf{K}(\mathcal{A})$ will denote the associated homotopy category, with its natural structure as a triangulated category. Morphisms in this category are denoted $\operatorname{Hom}_{\mathbf{K}(\mathcal{A})}(-,-)$. An object X in $\mathbf{K}(\mathcal{A})$ is *acyclic* if $H^*(X) = 0$, and the full subcategory of acyclic objects in $\mathbf{K}(\mathcal{A})$ is denoted $\mathbf{K}_{\mathrm{ac}}(\mathcal{A})$. A complex $X \in \mathbf{K}(\mathcal{A})$ is said to be *bounded above* if $X^i = 0$ for $i \gg 0$, and *bounded below* if $X^i = 0$ for $i \ll 0$.

In the sequel our focus in mostly on $\mathbf{K}(\operatorname{Inj} A)$, the homotopy category of injective modules, and its various subcategories; the analogous categories of projectives play a more subsidiary role. From work in [33, 35, 41], we know that the triangulated categories $\mathbf{K}(\operatorname{Inj} A)$ and $\mathbf{K}(\operatorname{Proj} A)$ are compactly generated since the ring A is Noetherian on both sides; the compact objects in these categories are described further below. Let $\mathbf{D}(\operatorname{Mod} A)$ denote the (full) derived category of A-modules and $\mathbf{q}\colon \mathbf{K}(\operatorname{Mod} A) \to \mathbf{D}(\operatorname{Mod} A)$ the localization functor; its kernel is $\mathbf{K}_{\mathrm{ac}}(\operatorname{Mod} A)$. We write $\mathbf{q}$ also for its restriction to the homotopy categories of injectives and projectives. These functors have adjoints:

$$\mathbf{K}(\operatorname{Inj} A) \underset{\mathbf{i}}{\overset{\mathbf{q}}{\rightleftarrows}} \mathbf{D}(\operatorname{Mod} A) \qquad \text{and} \qquad \mathbf{K}(\operatorname{Proj} A) \underset{\mathbf{q}}{\overset{\mathbf{p}}{\leftrightarrows}} \mathbf{D}(\operatorname{Mod} A)\,.$$

Our convention is to write the left adjoint above the corresponding right one. In what follows, it is convenient to conflate $\mathbf{i}$ and $\mathbf{p}$ with $\mathbf{i}\circ\mathbf{q}$ and $\mathbf{p}\circ\mathbf{q}$, respectively. The images of $\mathbf{i}$ and $\mathbf{p}$ are the K-injectives and K-projectives, respectively. Recall that an object X in $\mathbf{K}(\operatorname{Inj} A)$ is *K-injective* if $\operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(W, X) = 0$ for any acyclic complex W in $\mathbf{K}(\operatorname{Mod} A)$. We write $\mathbf{K}_{\mathrm{inj}}(A)$ for the full subcategory of $\mathbf{K}(\operatorname{Inj} A)$ consisting of K-injective complexes. The subcategory $\mathbf{K}_{\mathrm{proj}}(A) \subseteq \mathbf{K}(\operatorname{Proj} A)$ of K-projective complexes is defined similarly.

Compact objects. — Since A is Noetherian $\operatorname{Inj} A$ is closed under arbitrary direct sums, and hence so is the subcategory $\mathbf{K}(\operatorname{Inj} A)$ of $\mathbf{K}(\operatorname{Mod} A)$. As in any triangulated category with arbitrary direct sums, an object X in $\mathbf{K}(\operatorname{Inj} A)$ is *compact* if $\operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(X, -)$ commutes with direct sums. The compact objects in $\mathbf{K}(\operatorname{Inj} A)$ form a thick subcategory, denoted $\mathbf{K}^{\mathrm{c}}(\operatorname{Inj} A)$. The adjoint pair $(\mathbf{q}, \mathbf{i})$ above restricts to an equivalence of triangulated categories

$$\mathbf{K}^{\mathrm{c}}(\operatorname{Inj} A) \underset{\mathbf{i}}{\overset{\mathbf{q}}{\underset{\sim}{\rightleftarrows}}} \mathbf{D}^{\mathrm{b}}(\operatorname{mod} A)\,,$$

where $\mathbf{D}^{\mathrm{b}}(\operatorname{mod} A)$ denotes the bounded derived category of $\operatorname{mod} A$; see [35, Proposition 2.3] for a proof of this assertion. The corresponding identification of the compact objects in $\mathbf{K}(\operatorname{Proj} A)$ is a bit more involved and is due to Jørgensen [33, Theorem 3.2]. The assignment $M \mapsto \operatorname{Hom}_{A^{\mathrm{op}}}(\mathbf{p}M, A)$ induces an equivalence

$$\mathbf{D}^{\mathrm{b}}(\operatorname{mod} A^{\mathrm{op}})^{\mathrm{op}} \xrightarrow{\sim} \mathbf{K}^{\mathrm{c}}(\operatorname{Proj} A)\,.$$

See also [32], where these two equivalences are related. The formula below for computing morphisms from compacts in $\mathbf{K}(\operatorname{Inj} A)$ is useful in the sequel.

Lemma 2.1. — *For $C, X \in \mathbf{K}(\operatorname{Inj} A)$ with C compact, there is a natural isomorphism*

$$\operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(C, X) \cong H^0(\operatorname{Hom}_A(\mathbf{p}C, A) \otimes_A X)\,.$$

Proof. — Since C is compact its K-projective resolution $\mathbf{p}C$ is homotopy equivalent to a complex that is bounded above and consists of finitely generated projective A-modules. For each integer n, let $X(n)$ be the subcomplex $X^{\geqslant -n}$ of X. Since $X(n)$ is K-injective, the quasi-isomorphism $\mathbf{p}C \to C$ induces the one on the left

$$\operatorname{Hom}_A(C, X(n)) \xrightarrow{\sim} \operatorname{Hom}_A(\mathbf{p}C, X(n)) \xleftarrow{\sim} \operatorname{Hom}_A(\mathbf{p}C, A) \otimes_A X(n)\,.$$

The one on the right is the standard one and holds because of the aforementioned properties of $\mathbf{p}C$ and the fact that $X(n)$ is bounded below. One thus gets a canonical isomorphism

$$\operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(C, X(n)) \xrightarrow{\sim} H^0(\operatorname{Hom}_A(\mathbf{p}C, A) \otimes_A X(n))\,.$$

It is compatible with the inclusions $X(n) \subseteq X(n+1)$, so induces the isomorphism in the bottom row of the following diagram.

$$\begin{array}{ccc} \operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(C, \operatorname{hocolim}_{n\geqslant 0} X(n)) & \xrightarrow{\sim} & H^0(\operatorname{Hom}_A(\mathbf{p}C, A) \otimes_A \operatorname{hocolim}_{n\geqslant 0} X(n)) \\ \downarrow\wr & & \downarrow\wr \\ \operatorname{colim}_{n\geqslant 0} \operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(C, X(n)) & \xrightarrow{\sim} & \operatorname{colim}_{n\geqslant 0} H^0(\operatorname{Hom}_A(\mathbf{p}C, A) \otimes_A X(n))\,. \end{array}$$

The isomorphism on the left holds by the compactness of C, while the one on the right holds because $H^0(-)$ commutes with homotopy colimits. It remains to note that $\operatorname{hocolim}_{n\geqslant 0} X(n) = X$ in $\mathbf{K}(\operatorname{Inj} A)$. ☐

A recollement. — The functors $\mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A) \xrightarrow{\mathrm{incl}} \mathbf{K}(\operatorname{Inj} A) \xrightarrow{\mathbf{q}} \mathbf{D}(\operatorname{Mod} A)$ induce a recollement of triangulated categories

(1) $$\mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A) \underset{\overset{\mathbf{r}}{\twoheadleftarrow}}{\overset{\overset{\mathbf{s}}{\twoheadleftarrow}}{\xrightarrow{\mathrm{incl}}}} \mathbf{K}(\operatorname{Inj} A) \underset{\overset{\mathbf{i}}{\hookleftarrow}}{\overset{\overset{\mathbf{j}}{\hookleftarrow}}{\xrightarrow{\mathbf{q}}}} \mathbf{D}(\operatorname{Mod} A)\,.$$

The functor $\mathbf{i}$ is the one discussed above; it embeds $\mathbf{D}(\operatorname{Mod} A)$ as the homotopy category of K-injective complexes. The functor $\mathbf{r}$ thus has a simple description: there is an exact triangle

$$\mathbf{r}X \longrightarrow X \longrightarrow \mathbf{i}X \longrightarrow, \tag{2}$$

where the morphism $X \to \mathbf{i}X$ is the canonical one. Indeed, $\mathbf{r}X$ is evidently acyclic, and if W is in $\mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A)$, the induced map $\operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(W, \mathbf{r}X) \to \operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(W, X)$ is an isomorphism, for one has $\operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(W, \mathbf{i}X) = 0$.

The functor $\mathbf{j}\colon \mathbf{D}(\operatorname{Mod} A) \to \mathbf{K}(\operatorname{Inj} A)$ is fully faithful. The image of $\mathbf{j}$ equals the kernel of $\mathbf{s}$ and identifies with $\operatorname{Loc}(\mathbf{i}A)$, the localizing subcategory of $\mathbf{K}(\operatorname{Inj} A)$ generated by the injective resolution of A; see [35, Theorem 4.2]. One may think of $\mathbf{j}$ as the injective version of taking projective resolutions; see Lemma 2.5. To justify this claim takes preparation.

Lemma 2.2. — *Restricted to the subcategory* $\operatorname{Loc}(\mathbf{i}A)$ *of* $\mathbf{K}(\operatorname{Inj} A)$ *there is a natural isomorphism of functors* $\mathbf{r} \xrightarrow{\sim} \Sigma^{-1}\mathbf{si}$.

Proof. — Consider anew the exact triangle (2), but for X in $\operatorname{Loc}(\mathbf{i}A)$:

$$\mathbf{r}X \longrightarrow X \longrightarrow \mathbf{i}X \longrightarrow \Sigma\mathbf{r}X\,.$$

Apply $\mathbf{s}$ and remember that its kernel is $\operatorname{Loc}(\mathbf{i}A)$. □

Projective algebras. — In the remainder of this section, we assume that the ring A (which hitherto has been Noetherian on both sides) is also projective, as a module, over some central subring R. For the moment, the only role R plays is to allow for constructions of bimodule resolutions with good properties. Set $A^{\mathrm{ev}} := A \otimes_R A^{\mathrm{op}}$, the enveloping algebra of the R-algebra A, and set

$$E := \mathbf{i}_{A^{\mathrm{ev}}} A\,.$$

This is an injective resolution of A as a (left) module over A^{ev}. Since E is a complex of A-bimodules, for any complex X of A-modules, the right action of A on E induces a left A-action on $\operatorname{Hom}_A(E, X)$. The structure map $A \to E$ of bimodules induces a morphism of A-complexes

$$\operatorname{Hom}_A(E, X) \longrightarrow \operatorname{Hom}_A(A, X) \cong X \quad \text{for } X \in \mathbf{K}(\operatorname{Mod} A). \tag{3}$$

The computation below will be used often:

Lemma 2.3. — *The morphism in* (3) *is a quasi-isomorphism for* $X \in \mathbf{K}(\operatorname{Inj} A)$.

Proof. — By considering the mapping cone of $A \to E$, the desired statement reduces to: For any complex $W \in \mathbf{K}(\operatorname{Mod} A)$ that is acyclic and satisfies $W^i = 0$ for $i \ll 0$, one has $\operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(W, X) = 0$. Without loss of generality we can assume $W^i = 0$ for $i < 0$. Then one gets the first equality below

$$\operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(W, X) = \operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(W, X^{\geqslant -1}) = 0,$$

and the second one holds because $X^{\geqslant -1}$ is K-injective. □

Since A is projective as an R-module, A^{ev} is projective as an A-module both on the left and on the right. The latter condition implies, by adjunction, that as a complex of left A-modules E consists of injectives. In particular, for any projective A-module P, the A-complex $E \otimes_A P$ consists of injective modules. Thus, one has an exact functor

$$E \otimes_A -\colon \mathbf{K}(\operatorname{Proj} A) \longrightarrow \mathbf{K}(\operatorname{Inj} A)\,.$$

For each X in $\mathbf{K}(\operatorname{Inj} A)$, one has isomorphisms

$$\begin{aligned}\operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(E \otimes_A \mathbf{p}X, X) &\cong \operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(\mathbf{p}X, \operatorname{Hom}_A(E, X))\\ &\cong \operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(\mathbf{p}X, X)\,.\end{aligned}$$

The second isomorphism is a consequence of Lemma 2.3 and the K-projectivity of $\mathbf{p}X$. Thus, corresponding to the morphism $\mathbf{p}X \to X$, there is natural morphism

$$\pi(X)\colon E \otimes_A \mathbf{p}X \longrightarrow X \tag{4}$$

of complexes of A-modules.

Lemma 2.4. — *The morphism $\pi(X)$ in (4) is a quasi-isomorphism for each X.*

Proof. — Let $\eta\colon A \to E$ and $\varepsilon\colon \mathbf{p}X \to X$ denote the structure maps. These fit in the commutative diagram

$$\begin{array}{ccc} A \otimes_A \mathbf{p}X & \xrightarrow{\ \sim\ } & \mathbf{p}X \\ {\scriptstyle \eta\otimes_A \mathbf{p}X}\downarrow & & \downarrow{\scriptstyle \varepsilon} \\ E \otimes_A \mathbf{p}X & \xrightarrow{\ \pi(X)\ } & X\,. \end{array}$$

The map $\eta \otimes_A \mathbf{p}X$ is a quasi-isomorphism as η is one and $\mathbf{p}X$ is K-projective. Thus, $\pi(X)$ is a quasi-isomorphism. □

The stabilization functor. — The functor $\mathbf{s}\colon \mathbf{K}(\operatorname{Inj} A) \to \mathbf{K}_{\mathrm{ac}}(\operatorname{Inj} A)$ from (1) admits the following description in terms of its kernel, which uses the natural transformation $\pi\colon E \otimes_A \mathbf{p}(-) \to \mathrm{id}$ of functors on $\mathbf{K}(\operatorname{Inj} A)$ from (4).

Lemma 2.5. — *Each object X in $\mathbf{K}(\operatorname{Inj} A)$ fits into an exact triangle*

$$E \otimes_A \mathbf{p}X \xrightarrow{\pi(X)} X \longrightarrow \mathbf{s}X \longrightarrow\,,$$

and this yields a natural isomorphism $E \otimes_A \mathbf{p}X \xrightarrow{\sim} \mathbf{j}X$.

Proof. — Since $\pi(X)$ is a quasi-isomorphism, by Lemma 2.4, the complex $\mathbf{s}X$ is acyclic. In $\mathbf{K}(\operatorname{Proj} A)$, the complex $\mathbf{p}X$ is in $\operatorname{Loc}(A)$, and hence in $\mathbf{K}(\operatorname{Inj} A)$, the complex $E \otimes_A \mathbf{p}X$ is in $\operatorname{Loc}(E)$. It remains to observe that if $W \in \mathbf{K}(\operatorname{Inj} A)$ is acyclic, then $\operatorname{Hom}_{\mathbf{K}(\mathrm{A})}(E, W) = 0$ by Lemma 2.3. □