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SUPERCUSPIDAL REPRESENTATIONS OF $\mathrm{GL}_n(F)$ DISTINGUISHED BY A UNITARY INVOLUTION

by Jiandi Zou

Abstract. — Let F/F_0 be a quadratic extension of non-Archimedean locally compact fields of residue characteristic $p \neq 2$. Let R be an algebraically closed field of characteristic different from p. For π a supercuspidal representation of $G = \mathrm{GL}_n(F)$ over R and G^τ a unitary subgroup of G with respect to F/F_0, we prove that π is distinguished by G^τ, if and only if π is Galois invariant. When $R = \mathbb{C}$ and F is a p-adic field, this result was first a conjecture proposed by Jacquet and was proved in the 2010s by Feigon–Lapid–Offen by using global methods. Our proof is local and works for both complex representations and l-modular representations with $l \neq p$. We further study the dimension of $\mathrm{Hom}_{G^\tau}(\pi, 1)$ and show that it is at most 1.

Résumé (*Représentations supercuspidales de* $\mathrm{GL}_n(F)$ *distinguées par une involution unitaire*). — Soit F/F_0 une extension quadratique de corps localement compacts non archimédiens de caractéristique résiduelle $p \neq 2$. Soit R un corps algébriquement clos de caractéristique différente de p. Pour π une représentation supercuspidale de $G = \mathrm{GL}_n(F)$ sur R et G^τ un sous-groupe unitaire de G par rapport à F/F_0, on montre que π est distinguée par G^τ si et seulement si π est invariante galoisienne. Lorsque $R = \mathbb{C}$ et F est un corps p-adique, ce résultat d'abord sous la forme d'une conjecture proposée par Jacquet a été prouvé dans les années 2010 par Feigon-Lapid-Offen en utilisant des méthodes globales. Notre preuve est locale et fonctionne à la fois pour les représentations complexes et les représentations l-modulaires avec $l \neq p$. Nous étudions plus en détail la dimension de $\mathrm{Hom}_{G^\tau}(\pi, 1)$ et montrons qu'elle est au plus un.

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1. Introduction

Let F/F_0 be a quadratic extension of p-adic fields of residue characteristic p and let σ denote its non-trivial automorphism. For $G = \mathrm{GL}_n(F)$, let ε be a *hermitian matrix* in G, that is, $\sigma({}^t\varepsilon) = \varepsilon$ with ^{t} denoting the transpose of matrices. We define

$$\tau_\varepsilon(x) = \varepsilon\sigma({}^t x^{-1})\varepsilon^{-1},$$

for any $x \in G$, called a *unitary involution* on G. We fix $\tau = \tau_\varepsilon$ and we denote by G^τ the subgroup of G consisting of the elements fixed by τ, called the *unitary subgroup* of G with respect to τ. For π an irreducible smooth representation of G over $\mathbb{C}$, Jacquet proposed to study the space of G^τ-invariant linear forms on π, that is, the space

$$\mathrm{Hom}_{G^\tau}(\pi, 1).$$

When the space is non-zero, he called π *distinguished by* G^τ. For $n = 3$ and π supercuspidal, he proved in [26] by using global argument that π is distinguished by G^τ, if and only if π is σ-invariant, that is, $\pi^\sigma \cong \pi$, where $\pi^\sigma := \pi \circ \sigma$. Moreover, he showed that this space is of dimension 1 as a complex vector space when the condition above is satisfied. Moreover, in *ibid.*, he also sketched a similar proof when $n = 2$ and π is supercuspidal to give the same criterion of being distinguished and the same dimension 1 theorem. Based on these results, he conjectured that, in general, π is distinguished by G^τ, if and only if π is σ-invariant. Moreover, it is also interesting to determine the dimension of the space of G^τ-invariant linear forms that is not necessarily 1 in general. Under the assumption that π is σ-invariant and supercuspidal, Jacquet further conjectured that the dimension is 1.

In addition, an irreducible representation π of G is contained in the image of quadratic base change with respect to F/F_0, if and only if it is σ-invariant ([3]). Thus, for irreducible representations, the conjecture of Jacquet gives a connection between quadratic base change and G^τ-distinction.

Besides the special case mentioned above, the following two evidences also support the conjecture. First, we consider the analogue of the conjecture in the finite field case. For $\overline{\rho}$ an irreducible complex representation of $\mathrm{GL}_n(\mathbb{F}_{q^2})$, Gow [16] proved that $\overline{\rho}$ is distinguished by the unitary subgroup $\mathrm{U}_n(\mathbb{F}_q)$, if and only if $\overline{\rho}$ is isomorphic to its twist under the non-trivial element of $\mathrm{Gal}(\mathbb{F}_{q^2}/\mathbb{F}_q)$. Under this condition, he also showed that the space of $\mathrm{U}_n(\mathbb{F}_q)$-invariant linear forms is of dimension 1 as a complex vector space. In addition, Shintani [41] showed that there is a one-to-one correspondence between the set of irreducible representations of $\mathrm{GL}_n(\mathbb{F}_q)$ and that of Galois-invariant irreducible representations of $\mathrm{GL}_n(\mathbb{F}_{q^2})$, where the correspondence, called the *base change map*, is characterized by a trace identity. Thus, these two results relate the $\mathrm{U}_n(\mathbb{F}_q)$-distinction to the base change map. Finally, when $\overline{\rho}$ is generic and Galois-invariant, Anandavardhanan and Matringe [2] recently showed that the $\mathrm{U}_n(\mathbb{F}_q)$-average of the Bessel function of $\overline{\rho}$ on the Whittaker model as a $\mathrm{U}_n(\mathbb{F}_q)$-invariant linear form is non-zero. Since the space of $\mathrm{U}_n(\mathbb{F}_q)$-invariant linear forms is of dimension 1, their result gives us a concrete characterization of the space.

The other evidence for the Jacquet conjecture is its global analogue. We assume $\mathcal{K}/\mathcal{K}_0$ to be a quadratic extension of number fields and we denote by σ its non-trivial automorphism. We choose τ to be a unitary involution on $\mathrm{GL}_n(\mathcal{K})$, which also gives us an involution on $\mathrm{GL}_n(\mathbb{A}_\mathcal{K})$, still denoted by τ by abuse of notation, where $\mathbb{A}_\mathcal{K}$ denotes the ring of adèles of $\mathcal{K}$. We denote by $\mathrm{GL}_n(\mathcal{K})^\tau$ (or $\mathrm{GL}_n(\mathbb{A}_\mathcal{K})^\tau$) the unitary subgroup of $\mathrm{GL}_n(\mathcal{K})$ (or $\mathrm{GL}_n(\mathbb{A}_\mathcal{K})$) with respect to τ. For ϕ a cusp form of $\mathrm{GL}_n(\mathbb{A}_\mathcal{K})$, we define

$$\mathcal{P}_\tau(\phi) = \int_{\mathrm{GL}_n(\mathcal{K})^\tau \backslash \mathrm{GL}_n(\mathbb{A}_\mathcal{K})^\tau} \phi(h)dh$$

to be the *unitary period integral* of ϕ with respect to τ. We say that a cuspidal automorphic representation Π of $\mathrm{GL}_n(\mathbb{A}_\mathcal{K})$ is $\mathrm{GL}_n(\mathbb{A}_\mathcal{K})^\tau$-distinguished if there exists a cusp form in the space of Π such that $\mathcal{P}_\tau(\phi) \neq 0$. In the 1990s, Jacquet and Ye began to study the relation between $\mathrm{GL}_n(\mathbb{A}_\mathcal{K})^\tau$-distinction and global base change (see, for example, [28] when $n = 3$). For general n, Jacquet [27] showed that Π is contained in the image of the quadratic base change map (or equivalently Π is σ-invariant [3]) with respect to $\mathcal{K}/\mathcal{K}_0$, if and only if there exists a unitary involution τ such that Π is G^τ-distinguished. This result may be viewed as the global version of the Jacquet conjecture for supercuspidal representations.

In fact, for the special case of the Jacquet conjecture in [26], Jacquet used the global analogue of the same conjecture and the relative trace formula to finish the proof. To say it simply, he first proved the global analogue of the conjecture. Then he used the relative trace formula to write a non-zero unitary period integral as the product of its local components at each place of $\mathcal{K}_0$, where each local component characterizes the distinction of the local component of Π with respect to the corresponding unitary group over local fields. When π is σ-invariant, he chose Π to be a σ-invariant cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_\mathcal{K})$ and v_0 to be a non-Archimedean place of $\mathcal{K}_0$, such that $(G^\tau, \pi) = (\mathrm{GL}_n(\mathcal{K}_{v_0})^\tau, \Pi_{v_0})$. Then the product decomposition leads to the proof of the "if" part of the conjecture. The "only if" part of the conjecture, which will be discussed in Section 4, requires the application of a globalization theorem. His method was generalized by Feigon–Lapid–Offen in [14] to general n and more general families of representations. They showed that the Jacquet conjecture works for generic representations of G. Moreover, they were able to give a lower bound for the dimension of $\mathrm{Hom}_{G^\tau}(\pi, 1)$ and they further conjectured that the inequality they gave is actually an equality. Finally, Beuzart-Plessis recently