

quatrième série - tome 49 fascicule 1 janvier-février 2016

*ANNALES
SCIENTIFIQUES
de
L'ÉCOLE
NORMALE
SUPÉRIEURE*

Erwan ROUSSEAU

Hyperbolicity, automorphic forms and Siegel modular varieties

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Annales Scientifiques de l'École Normale Supérieure

Publiées avec le concours du Centre National de la Recherche Scientifique

Responsable du comité de rédaction / *Editor-in-chief*

Antoine CHAMBERT-LOIR

Publication fondée en 1864 par Louis Pasteur

Continuée de 1872 à 1882 par H. SAINTE-CLAIRE DEVILLE
de 1883 à 1888 par H. DEBRAY
de 1889 à 1900 par C. HERMITE
de 1901 à 1917 par G. DARBOUX
de 1918 à 1941 par É. PICARD
de 1942 à 1967 par P. MONTEL

Comité de rédaction au 1^{er} janvier 2016

N. ANANTHARAMAN I. GALLAGHER
P. BERNARD B. KLEINER
E. BREUILLARD E. KOWALSKI
R. CERF M. MUSTAȚĂ
A. CHAMBERT-LOIR L. SALOFF-COSTE

Rédaction / *Editor*

Annales Scientifiques de l'École Normale Supérieure,
45, rue d'Ulm, 75230 Paris Cedex 05, France.
Tél. : (33) 1 44 32 20 88. Fax : (33) 1 44 32 20 80.
annales@ens.fr

Édition / *Publication*

Société Mathématique de France
Institut Henri Poincaré
11, rue Pierre et Marie Curie
75231 Paris Cedex 05
Tél. : (33) 01 44 27 67 99
Fax : (33) 01 40 46 90 96

Abonnements / *Subscriptions*

Maison de la SMF
Case 916 - Luminy
13288 Marseille Cedex 09
Fax : (33) 04 91 41 17 51
email : smf@smf.univ-mrs.fr

Tarifs

Europe : 515 €. Hors Europe : 545 €. Vente au numéro : 77 €.

© 2016 Société Mathématique de France, Paris

En application de la loi du 1^{er} juillet 1992, il est interdit de reproduire, même partiellement, la présente publication sans l'autorisation de l'éditeur ou du Centre français d'exploitation du droit de copie (20, rue des Grands-Augustins, 75006 Paris).

All rights reserved. No part of this publication may be translated, reproduced, stored in a retrieval system or transmitted in any form or by any other means, electronic, mechanical, photocopying, recording or otherwise, without prior permission of the publisher.

ISSN 0012-9593

Directeur de la publication : Marc Peigné
Périodicité : 6 n^{os} / an

HYPERBOLICITY, AUTOMORPHIC FORMS AND SIEGEL MODULAR VARIETIES

BY ERWAN ROUSSEAU

ABSTRACT. – We study the hyperbolicity of compactifications of quotients of bounded symmetric domains by arithmetic groups. We prove that, up to a finite étale cover, they are Kobayashi hyperbolic modulo the boundary. Applying our techniques to Siegel modular varieties, we improve some former results of Nadel on the non-existence of certain level structures on Abelian varieties over complex function fields.

RÉSUMÉ. – Nous étudions l’hyperbolicité des compactifications de quotients de domaines symétriques bornés par des groupes arithmétiques. Nous prouvons, qu’à un revêtement étale fini près, ils sont hyperboliques au sens de Kobayashi modulo le bord. En appliquant ces techniques aux variétés de Siegel modulaires, nous améliorons des résultats antérieurs de Nadel sur la non-existence de certaines structures de niveau pour les variétés abéliennes définies sur des corps de fonctions.

1. Introduction

Any complex space Y can be equipped with an intrinsic pseudo-distance d_Y , the Kobayashi pseudo-distance [6]. It is the largest pseudo-distance such that every holomorphic map $f : \Delta \rightarrow Y$ from the unit disc, equipped with the Poincaré metric, is distance decreasing.

Recall [6] that a complex space Y is said to be (Kobayashi) hyperbolic modulo $W \subset Y$ if for every pair of distinct points p, q of Y we have $d_Y(p, q) > 0$ unless both are contained in W . In the case where Y is compact and $W = \emptyset$, Brody’s lemma gives a criterion for hyperbolicity: Y is hyperbolic if and only if there are no curves $f : \mathbb{C} \rightarrow Y$.

In the case where $W \neq \emptyset$, we do not know an analogue of Brody’s criterion for hyperbolicity modulo W . It is not known whether Y is necessarily hyperbolic modulo W if it is

Partially supported by the ANR project “POSITIVE”, ANR-2010-BLAN-0119-01. This work has been carried out in the framework of the Labex Archimède (ANR-11-LABX-0033) and of the A*MIDEX project (ANR-11-IDEX-0001-02), funded by the “Investissements d’Avenir” French Government programme managed by the French National Research Agency (ANR).

Brody hyperbolic modulo W i.e., every entire curve $f : \mathbb{C} \rightarrow Y$ has image in W . The reason is that the classical Brody's lemma does not provide any information on the image of the entire curve.

In this article, we investigate hyperbolic properties of compactifications of quotients of bounded symmetric domains by arithmetic groups. It is well known that such quotients may be far from being hyperbolic, as rational Hilbert modular surfaces show.

Let $X := \mathbb{B}/\Gamma$ be a quotient of a bounded symmetric domain by an arithmetic subgroup $\Gamma \subset \text{Aut}(\mathbb{B})$. By passing to a subgroup of finite index if necessary, we will always suppose that Γ is neat [2]. In particular, X admits a smooth toroidal compactification \overline{X} where $D = \overline{X} \setminus X$ is a normal crossings divisor [1]. Moreover, \overline{X} may be chosen to be projective [1].

We obtain the following result

THEOREM 1.1. – *Let $X := \mathbb{B}/\Gamma$ be a quotient of a bounded symmetric domain by a neat arithmetic subgroup $\Gamma \subset \text{Aut}(\mathbb{B})$. Then there exists a finite étale cover $X_1 = \mathbb{B}/\Gamma_1$ determined by a subgroup $\Gamma_1 \subset \Gamma$ such that $\overline{X_1}$, a smooth projective compactification of X_1 , is Kobayashi hyperbolic modulo $D_1 := \overline{X_1} \setminus X_1$.*

As a corollary, we obtain a new simple proof of the main result of [8], where it is proved that $\overline{X_1}$ is Brody hyperbolic modulo D_1 : the image $f(\mathbb{C})$ of any non-constant holomorphic map $f : \mathbb{C} \rightarrow \overline{X_1}$ is contained in D_1 .

While Nadel's proof [8] was based on Nevanlinna theory, our proof consists in using the fundamental distance decreasing property of the Kobayashi pseudo-metric mentioned above. To prove Theorem 1.1, we will construct pseudo-distances on $\overline{X_1}$ for which holomorphic maps from the unit disc are distance decreasing.

As an application, we study Siegel modular varieties from this point of view. Their geometry has attracted a lot of attention (see [4] for a survey). In particular, the birational geometry of \mathcal{A}_g and $\mathcal{A}_g(n)$, the moduli spaces of principally polarized Abelian varieties with level structures, has been extensively investigated. The Kodaira dimension of compactifications $\overline{\mathcal{A}_g(n)}$ has been studied by Tai, Freitag, Mumford, Hulek and others, proving the following result.

THEOREM 1.2 (Tai, Freitag, Mumford, Hulek). – *$\overline{\mathcal{A}_g(n)}$ is of general type for the following values of g and $n \geq n_0$:*

g	1	2	3	4	5	6	≥ 7
n_0	7	4	3	2	2	2	1

Therefore it is very natural to study hyperbolicity of these spaces in the light of the Green-Griffiths-Lang conjecture

CONJECTURE 1.3 (Green-Griffiths, Lang). – *Let X be a projective variety of general type. Then there exists a proper algebraic subvariety $Y \subset X$ such that X is hyperbolic modulo Y .*

In this setting, using the theory of automorphic forms, we can apply the same strategy as above and obtain

THEOREM 1.4. – *$\overline{\mathcal{A}_g(n)}$ is hyperbolic modulo $D := \overline{\mathcal{A}_g(n)} \setminus \mathcal{A}_g(n)$ if $n > 6g$.*

As a corollary we have

THEOREM 1.5. – *If A is a principally polarized Abelian variety of dimension $g \geq 1$ which is defined and non-constant over k , a complex function field of genus ≤ 1 , then A does not admit a level- n structure for $n > 6g$.*

In particular, this improves the previous bound, $n \geq \max(\frac{1}{2}g(g + 1), 28)$, obtained by Nadel in [8].

Acknowledgments. – The author is grateful to Carlo Gasbarri and Xavier Roulleau for several interesting and fruitful discussions and remarks. He also thanks CNRS for the opportunity to spend a semester in Montréal at the UMI CNRS-CRM and the hospitality of UQAM-Cirget where part of this work was done.

2. Distance decreasing pseudo-distances

Let $X := \mathbb{B}/\Gamma$ be a quotient of a bounded symmetric domain by an arithmetic subgroup $\Gamma \subset \text{Aut}(\mathbb{B})$. We first give a criterion to ensure the existence of a pseudo-distance \tilde{g} on \overline{X} non-degenerate at a given $x \in X$ and satisfying the distance decreasing property: every holomorphic map $f : (\Delta, g_P) \rightarrow (\overline{X}, \tilde{g})$ from the unit disc, equipped with the Poincaré metric g_P , is distance decreasing i.e.,

$$f^*\tilde{g} \leq g_P.$$

Let g denote the Bergman metric on \mathbb{B} satisfying the Kähler-Einstein property $\text{Ric}(g) = -g$. Let $\gamma > 0$ be a rational number such that g has holomorphic sectional curvature $\leq -\gamma$. We want to construct a continuous function $\psi : \overline{X} \rightarrow \mathbb{R}^+$ such that $\psi \cdot g$ defines a pseudo-metric on \overline{X} .

THEOREM 2.1. – *Let $x \in X$ and suppose there is a section $s \in H^0(\overline{X}, l(K_{\overline{X}} + D))$ such that*

- $s(x) \neq 0$
- s vanishes on D with multiplicity $m > \frac{l}{\gamma}$.

Then there exists a pseudo-metric \tilde{g} on \overline{X} non-degenerate at x satisfying the distance decreasing property. In particular, the Kobayashi pseudo-metric on \overline{X} is non-degenerate at x .

We will prove this theorem in 3 steps.

From [7], we know that the Bergman metric induces a good singular metric $h := (\det g)^{-1}$ on $K_{\overline{X}} + D$. Let $x \in X$ and s be as given above. Let $0 < \epsilon$ be such that

$$\gamma - \epsilon > \frac{l}{m}.$$

We define

$$\psi := \|s\|_h^{\frac{2(\gamma-\epsilon)}{l}}.$$

Despite the singularities of h , we have

PROPOSITION 2.2. – *ψ is a continuous function on \overline{X} vanishing on D .*