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Christophe GARBAN & Hugo VANNEUVILLE

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## EXCEPTIONAL TIMES FOR PERCOLATION UNDER EXCLUSION DYNAMICS

BY CHRISTOPHE GARBAN AND HUGO VANNEUVILLE

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ABSTRACT. – We analyze in this paper a conservative analog of the celebrated model of *dynamical percolation* introduced by Häggström, Peres and Steif in [10]. It is simply defined as follows: start with an initial percolation configuration  $\omega(t = 0)$ . Let this configuration evolve in time according to a simple exclusion process with symmetric kernel  $K(x, y)$ . We start with a general investigation (following [10]) of this dynamical process  $t \mapsto \omega_K(t)$  which we call *K-exclusion dynamical percolation*. We then proceed with a detailed analysis of the planar case at the critical point (both for the triangular grid and the square lattice  $\mathbb{Z}^2$ ) where we consider the power-law kernels  $K^\alpha$

$$K^\alpha(x, y) \propto \frac{1}{\|x - y\|_2^{2+\alpha}}.$$

We prove that if  $\alpha > 0$  is chosen small enough, there exist *exceptional times*  $t$  for which an infinite cluster appears in  $\omega_{K^\alpha}(t)$ . (On the triangular grid, we prove that this holds for all  $\alpha < \alpha_0 = \frac{217}{816}$ .) The existence of such exceptional times for standard i.i.d. dynamical percolation (where sites evolve according to independent Poisson point processes) goes back to the work by Schramm-Steif in [25]. In order to handle such a *K-exclusion* dynamics, we push further the spectral analysis of *exclusion noise sensitivity* which has been initiated in [3]. (The latter paper can be viewed as a conservative analog of the seminal paper by Benjamini-Kalai-Schramm [1] on i.i.d. noise sensitivity.) The case of a nearest-neighbor simple exclusion process, corresponding to the limiting case  $\alpha = +\infty$ , is left widely open.

RÉSUMÉ. – Cet article porte sur une version conservative du modèle de la *percolation dynamique* introduit par Häggström, Peres et Steif dans [10]. Le modèle se définit simplement de la façon suivante : on tire une configuration de percolation initiale  $\omega(t = 0)$ . Puis, on fait évoluer cette configuration selon un processus d'exclusion simple de noyau symétrique  $K(x, y)$ . On commence par une étude générale (en suivant [10]) du processus  $t \mapsto \omega_K(t)$  que l'on appelle *percolation dynamique sous K-exclusion*. Nous analysons ensuite de façon détaillée le cas bi-dimensionnel au point critique (à la fois pour le réseau triangulaire et pour le réseau  $\mathbb{Z}^2$ ) pour des noyaux en loi de puissance  $K^\alpha$

$$K^\alpha(x, y) \propto \frac{1}{\|x - y\|_2^{2+\alpha}}.$$

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Nous montrons que si l'exposant  $\alpha > 0$  est suffisamment petit, il existe des *temps exceptionnels*  $t$  pour lesquels une composante connexe infinie se forme dans  $\omega_{K^\alpha}(t)$ . (Pour la percolation par site sur réseau triangulaire, on montre que cela se produit pour tout  $\alpha < \alpha_0 = \frac{217}{816}$ ). L'existence de tels temps exceptionnels pour la percolation dynamique standard i.i.d. (où les sites évoluent selon des processus de Poisson indépendants) remonte au travail de Schramm-Steif [25]. Afin de contrôler la dynamique ci-dessus du type  $K$ -exclusion, on approfondit l'analyse spectrale de la *sensibilité au bruit sous exclusion* initiée dans le travail [3]. (Travail qui est en quelque sorte l'analogue conservatif du papier précurseur par Benjamini-Kalai-Schramm [1] sur la sensibilité au bruit i.i.d.). Le cas du processus d'exclusion simple au plus proche voisin, correspondant au cas limite  $\alpha = +\infty$ , reste entièrement ouvert.

## 1. Introduction

### 1.1. Dynamical percolation

We consider bond percolation on an infinite, countable, connected, locally finite graph  $G = (V, E)$ . We write  $\mathbb{P}_p$  for the probability measure of (bond) *percolation of parameter*  $p$  on  $G$  i.e., the probability measure on  $\Omega = \{-1, 1\}^E$  obtained by declaring each edge *open* with probability  $p$  and *closed* with probability  $1 - p$ , independently of the others (1 means open and  $-1$  means closed). More formally,  $\mathbb{P}_p$  is the product measure  $(p\delta_1 + (1 - p)\delta_{-1})^{\otimes E}$  on  $\Omega$  equipped with the product  $\sigma$ -algebra. An element  $\omega \in \Omega$  is called a *percolation configuration*. Moreover, a connected component of the graph obtained by keeping only the open edges is called a *cluster*. It is a simple consequence of Kolmogorov's 0-1 law that, for each  $p$ ,  $\mathbb{P}_p[\exists \text{ an infinite cluster}] \in \{0, 1\}$ . Moreover, it is well known (see for instance [9] or [2]) that there exists a *critical point*  $p_c = p_c(G) \in [0, 1]$  such that:

$$\begin{aligned} \forall p \in [0, p_c), \mathbb{P}_p[\exists \text{ an infinite cluster}] &= 0, \\ \forall p \in (p_c, 1], \mathbb{P}_p[\exists \text{ an infinite cluster}] &= 1. \end{aligned}$$

The most studied model is bond percolation on the Euclidean lattice  $\mathbb{Z}^d$ ,  $d \geq 2$ . For this model, it is known that  $p_c = p_c(d) \in (0, 1)$ . In other words, there exists a phase transition. Moreover, it is a celebrated theorem by Kesten [15] that  $p_c(2) = 1/2$  and it is conjectured that, for any  $d \geq 2$ ,  $\mathbb{P}_{p_c}[\exists \text{ an infinite cluster}] = 0$ . This last property has been proved for  $d = 2$  ([13]) and  $d \geq 11$  (see [12, 4]).

In [10], Häggström, Peres and Steif define and study the model of *dynamical* (bond) *percolation* (this model was invented independently by Benjamini). Dynamical percolation of parameter  $p \in [0, 1]$  is defined very easily as follows: we sample a percolation configuration  $\omega(0)$  according to some initial law and we then let evolve each edge independently of each other according to Poisson point processes: at rate one, the states of edges are resampled using  $p\delta_1 + (1 - p)\delta_{-1}$ . We obtain this way a càdlàg Markov process  $(\omega(t))_{t \geq 0}$  on the space  $\Omega$  (seen as the compact metric product space) with  $\mathbb{P}_p$  as (unique) invariant probability measure. The main question is whether, if  $\omega(0) \sim \mathbb{P}_p^{(1)}$ , there exist *exceptional times* for which the percolation configuration is very atypical. Exceptional times are defined as follows: if  $\mathbb{P}_p[\exists \text{ an infinite cluster}] = 0$ , then an exceptional time is a time for which there

<sup>(1)</sup> Where  $X \sim P$  means that  $P$  is the distribution of the random variable  $X$ .

is an infinite cluster. On the other hand, if  $\mathbb{P}_p [\exists \text{ an infinite cluster}] = 1$ , then it is a time for which there is no infinite cluster.

From now on, we assume that  $\omega(0) \sim \mathbb{P}_p$ . Since  $\mathbb{P}_p$  is an invariant measure, then (by Fubini) a.s. Leb-a.e. there is no exceptional time (where Leb is the Lebesgue measure on  $\mathbb{R}_+$ ). This does not imply that a.s. there does not exist any exceptional time. However, this is the case away from the critical point: the authors of [10] have proved that, for any graph  $G$ , if  $p \neq p_c$  then a.s. there is no exceptional time (see their Proposition 1.1).

The case  $p = p_c$  is in general much more difficult. First, let us note that, for bond percolation on the Euclidean lattice  $\mathbb{Z}^d$ , this is for now interesting only for  $d = 2$  and  $d \geq 11$  since these are the only dimensions for which we know what happens at criticality. For  $d \geq 11$ , thanks to a result proved in [12] for  $d \geq 19$  (and extended very recently to  $d \geq 11$  in [4]), the authors of [10] have proved that, even at criticality, a.s. there is no exceptional time (see their Theorem 1.3). However, for  $d = 2$ , the following is proved in [5] (Theorem 1.4):

for dynamical bond percolation on  $\mathbb{Z}^2$ , a.s. there are exceptional times if  $p = p_c = 1/2$ .

Such a result had been proved earlier in [25] for the model of *site percolation on the triangular lattice*. Let  $\mathbb{T}$  denote the (planar) triangular lattice and let  $\mathbb{P}_p$  denote the probability measure of site percolation on  $\mathbb{T}$  (this is the analogous model where the sites—i.e., the vertices of  $\mathbb{T}$ —are open or closed; in this context a cluster is a connected component of the graph obtained by keeping only the open sites). Kesten's work also implies that  $p_c = 1/2$  for this model. Of course, one can define dynamical site percolation on  $\mathbb{T}$  in the same way as for dynamical bond percolation i.e., by associating exponential clocks to the sites of  $\mathbb{T}$ . Much more is known for site percolation on  $\mathbb{T}$  than for bond percolation on  $\mathbb{Z}^2$ . Indeed conformal invariance (as the mesh goes to zero) has been proved by Smirnov in [26], and the exact value of several critical exponents (see Subsection 2.1) has been derived in [18, 27] using the *Schramm Loewner Evolution (SLE)* processes introduced by Schramm. Using the knowledge of these critical exponents, the following is proved in [25] (Theorem 1.3):

For dynamical site percolation on  $\mathbb{T}$ , a.s. there are exceptional times if  $p = p_c = 1/2$ .

Finally, let us mention that in [5] it is shown that, for critical site percolation on  $\mathbb{T}$ , the Hausdorff dimension of the set of exceptional times is a.s.  $31/36$ . For other results, see for instance [11] where the authors show that typical exceptional times are intimately related to the so-called Incipient Infinite Cluster introduced by Kesten.

In both [25] and [5], the main methods are related to the theory of *Fourier decomposition of Boolean functions*. In the present paper, we will also rely extensively on such tools, see Subsections 2.3 and 2.4.

## 1.2. Percolation under exclusion dynamics

We study in this paper the same question of existence of exceptional times but with a different underlying dynamical process: we let the configuration evolve according to a *symmetric exclusion process*. Percolation evolving according to an exclusion process has already been studied by Broman, the first author and Steif in [3] where the authors introduce and study the notion of *exclusion sensitivity*. (We will say more about this notion in Section 2, see also [22, 21].) To define and study a symmetric exclusion process (which is a Feller Markov