ISSN 0012-9593

quatrième série - tome 53 fascicule 2 mars-avril 2020

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

Thomas HAETTEL

Hyperbolic rigidity of higher rank lattices

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Annales Scientifiques de l'École Normale Supérieure

Publiées avec le concours du Centre National de la Recherche Scientifique

Rédaction / *Editor*

Annales Scientifiques de l'École Normale Supérieure,
45, rue d'Ulm, 75230 Paris Cedex 05, France.
Tél. : (33) 1 44 32 20 88. Fax : (33) 1 44 32 20 80.
annales@ens.fr

Édition et abonnements / *Publication and subscriptions*

Société Mathématique de France
Case 916 - Luminy
13288 Marseille Cedex 09
Tél. : (33) 04 91 26 74 64
Fax : (33) 04 91 41 17 51
email : abonnements@smf.emath.fr

Tarifs

Abonnement électronique : 420 euros.
Abonnement avec supplément papier :
Europe : 551 €. Hors Europe : 620 € ($ 930). Vente au numéro : 77 €.

ISSN 0012-9593 (print) 1873-2151 (electronic)

Directeur de la publication : Stéphane Seuret
Périodicité : 6 nos / an

Ann. Scient. Éc. Norm. Sup.
4^{e} série, t. 53, 2020, p. 439 à 468

HYPERBOLIC RIGIDITY OF HIGHER RANK LATTICES

BY Thomas HAETTEL
Appendix by Vincent Guirardel and Camille Horbez

ABSTRACT. – We prove that any action of a higher rank lattice on a Gromov-hyperbolic space is elementary. More precisely, it is either elliptic or parabolic. This is a large generalization of the fact that any action of a higher rank lattice on a tree has a fixed point. A consequence is that any quasi-action of a higher rank lattice on a tree is elliptic, i.e., it has Manning's property (QFA). Moreover, we obtain a new proof of the theorem of Farb-Kaimanovich-Masur that any morphism from a higher rank lattice to a mapping class group has finite image, without relying on the Margulis normal subgroup theorem nor on bounded cohomology. More generally, we prove that any morphism from a higher rank lattice to a hierarchically hyperbolic group has finite image. In the appendix, Vincent Guirardel and Camille Horbez deduce rigidity results for morphisms from a higher rank lattice to various outer automorphism groups.

RÉSUMÉ. – Nous montrons que toute action d'un réseau de rang supérieur sur un espace Gromov-hyperbolique est élémentaire. Plus précisément, toute action est elliptique ou parabolique. Ce résultat est une large généralisation du fait que toute action d'un réseau de rang supérieur sur un arbre a un point fixe. Une conséquence est que toute quasi-action d'un réseau de rang supérieur sur un arbre est elliptique, autrement dit il a la propriété (QFA) de Manning. De plus, nous obtenons une preuve nouvelle du théorème de Farb-Kaimanovich-Masur disant que tout morphisme d'un réseau de rang supérieur vers le groupe modulaire d'une surface est d'image finie, sans avoir recours au théorème du sous-groupe normal de Margulis ni à la cohomologie bornée. Enfin, nous montrons que tout morphisme d'un réseau de rang supérieur vers un groupe hiérarchiquement hyperbolique est d'image finie. Dans l'appendice, Vincent Guirardel et Camille Horbez déduisent des résultats de rigidité pour des morphismes de réseaux de rang supérieur à valeurs dans divers groupes d'automorphismes extérieurs.

Introduction

Higher rank semisimple algebraic groups over local fields, and their lattices, are well-known to enjoy various rigidity properties. The main idea is that they cannot act on any

 doi:10.24033/asens.2425

other space than the ones naturally associated to the Lie group. This is reflected notably in the Margulis superrigidity theorem, and is also the motivating idea of Zimmer's program.

Concerning the rigidity of isometric actions, the most famous example is Kazhdan's property (T), which tells us that higher rank lattices cannot act by isometries without fixed point on Hilbert spaces. In fact, property (T) also implies such a fixed point property for some L^p spaces (see [2]), for trees (see [48]), and more generally for metric median spaces (such as CAT(0) cube complexes, see [14]).

Property (T) is also satisfied notably by hyperbolic quaternionic lattices and by some random hyperbolic groups (see [51]). There have been various strengthenings of property (T), which are all satisfied by higher rank lattices but not by hyperbolic groups, which imply fixed point properties for various actions on various Banach spaces (see for instance [34] and [46]).

Since Gromov-hyperbolic spaces play a central role in geometric group theory, understanding actions of higher rank lattices on Gromov-hyperbolic spaces is an extremely natural question. There are several partial answers to that question, for instance any action on a tree or on a symmetric space of rank 1 has a fixed point. Manning proved that, for $\mathrm{SL}(n,\mathbb{Z})$ with $n \geqslant 3$ and some other boundedly generated groups, any action on quasi-tree is bounded (see [42]). Using V. Lafforgue's strengthened version of property (T) (see [34], [38], [33]), one deduces that if Γ is a cocompact lattice in a higher rank semisimple algebraic group, then any action of Γ by isometries on a uniformly locally finite Gromov-hyperbolic space is bounded.

The main purpose of this article is to prove the following.

Theorem A. – Let Γ be a lattice (in a product) of higher rank almost simple connected algebraic groups with finite centers over a local field. Then any action of Γ by isometries on a Gromov-hyperbolic metric space is elementary. More precisely, it is either elliptic or parabolic.

Remarks. –

- This result has also been announced by Bader and Furman, as it should be a consequence of their deep work on rigidity and boundaries (see notably [1, Theorem 4.1] for convergence actions of lattices). However, the techniques are essentially different: Bader and Furman use a lot of ergodic theory, while in this article we use very little of it, and focus mostly on the asymptotic geometry of lattices and buildings, making a crucial use of medians.
- One should note that the hyperbolic space in the theorem is not assumed to be locally compact, nor the action is assumed to satisfy any kind of properness.
- Note that most rigidity results conclude to the boundedness of orbits. Since any finitely generated group has a metrically proper, parabolic action on a hyperbolic space (locally infinite in general), one needs to include those parabolic actions (see for instance [26]).
- Even though they do not appear in the statement, the theory of coarse median spaces developed by Bowditch (see [8]) plays a crucial role in the proof.
- In the theorem, we have to assume that each almost simple factor has higher rank. Our methods use the induction to the ambient group, so we cannot study irreducible lattices in products of rank 1 groups. However, in this case, Bader and Furman can prove the following: for any irreducible lattice in a product of at least two groups, any isometric

action on a hyperbolic space X without bounded orbits in X or finite orbits in ∂X, there is a closed subset of the boundary on which the action extends to one factor.

- Whereas most rigidity results concerning higher rank lattices use bounded cohomology, Margulis superridigidity or normal subgroup theorems or V. Lafforgue's strengthenings of property (T), our proof uses really new ingredients, and in particular median spaces.

In [42], Manning was motivated by the question of quasi-actions of groups of trees. For the precise definition of a quasi-action, we refer to Section 5. Manning proved that any quasi-action of $SL(n,\mathbb{Z})$, for $n \geqslant 3$, on a tree is elliptic (or more generally $SL(n,\mathcal{O})$, where $\mathcal{O}$ is the integer ring of a number field, see [42, Corollary 4.5]). Manning used notably that $SL(n,\mathbb{Z})$ is boundedly generated by elementary matrices, which is not true for more general lattices. A straightforward consequence of Theorem A is the following generalization of Manning's result.

Corollary B. – Let Γ be as in Theorem A. Then any quasi-action of Γ by isometries on a tree is elliptic. In other words, Γ has Manning's property (QFA).

Another major consequence of Theorem A is another proof of the following.

Corollary C (Farb-Kaimanovich-Masur [17], [30]). – Let Γ be as in Theorem A, and let $S_{g,p}$ be a closed surface of genus g with p punctures. Then any morphism $\Gamma \to MCG(S_{g,p})$ has finite image.

The proof of Farb, Masur and Kaimanovich relies notably on the very deep Margulis normal subgroup theorem. Our purpose here is to give a proof as simple as possible, and we will not rely on any such deep theorem in the uniform case, and in the non-uniform one case we will use Margulis arithmeticity theorem only to ensure that the associated cocycle is integrable. In particular, in the proof of Corollary C, we will not even use Burger-Monod's result that higher rank lattices do not have unbounded quasi-morphisms. We will simply use the fact that higher rank lattices do not surject onto $\mathbb{Z}$ (it is a direct consequence of property (T)) and use the weaker form of Theorem A stating that every action of a higher rank lattice on a hyperbolic space is elementary.

In fact, we can also study the more general class of hierarchically hyperbolic groups. They have been defined and studied in several articles (see [5], [6], [4], [16]), and since the definition is technical and irrelevant for the rest of the article, we refer to these articles for the precise definitions and main results. Roughly speaking, hierarchically hyperbolic spaces are metric spaces with a nice collection of projections to hyperbolic spaces, organized with some hierarchical structure. Notable examples of hierarchically hyperbolic groups include hyperbolic groups, mapping class groups, right-angled Artin groups, and they are stable under relative hyperbolicity. Applying the exact same proof as in Corollary C yields the following more general result.

Corollary D. – Let Γ be as in Theorem A, and let G be a hierarchically hyperbolic group. Then any morphism $\Gamma \to G$ has finite image.