ADVANCED TOPICS IN RANDOM MATRICES

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ADVANCED TOPICS IN RANDOM MATRICES

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Abstract. **–** This book provides three accessible panoramas and syntheses on advanced topics in random matrix theory:

— local semicircle law for Wigner matrices, and applications to eigenvectors delocalization, rigidity of eigenvalues, and fourth moment theorem;
— spectrum of random graphs, recent advances on eigenvalues and eigenvectors, and open problems;
— deformed random matrices and free probability, unified understanding of various asymptotic phenomena such as spectral measure description, localization and fluctuations of extremal eigenvalues, eigenvectors behavior.

Résumé **(Thèmes avancés en matrices aléatoires). –** Ce livre propose trois panoramas et synthèses accessibles sur des thèmes avancés en matrices aléatoires:

— loi du semicercle locale pour les matrices de Wigner, et application à la délocalisation des vecteurs propres, à la rigidité des valeurs propres, ainsi qu'au théorème des quatre moments;
— spectres de graphes aléatoires, avancées récentes sur les valeurs propres et les vecteurs propres, et problèmes ouverts;
— matrices aléatoires déformées et probabilités libres, compréhension unifiée de divers phénomènes asymptotiques comme la description de la mesure spectrale, la localisation et les fluctuation des valeurs propres extrêmes, et le comportement des vecteurs propres.

CONTENTS

FOREWORD

This book provides three accessible panoramas and syntheses on recent, advanced hot topics in random matrix theory: local semicircle law for Wigner matrices, spectrum of random graphs, and deformed random matrices and free probability. These three chapters take their roots in the conference *États de la Recherche en Matrices Aléatoires* [1], held during the period December 1–4, 2014 in Institut Henri Poincaré, Paris, and co-organized by Florent Benaych-Georges (Université Paris-Descartes), Djalil Chafaï (Université Paris-Dauphine), Sandrine Péché (Université Paris-Diderot), and Béatrice de Tilière (Université Pierre et Marie Curie). The minicourses speakers were Charles Bordenave (CNRS & Université de Toulouse), Alexei Borodin (Massachusetts Institute of Technology), and Antti Knowles (Swiss Federal Institute of Technology in Zurich), while the complementary talks were given by Paul Bourgade (New York University), Cédric Boutillier (Université Pierre et Marie Curie), and Mireille Capitaine (CNRS & Université de Toulouse).

This conference belongs to the series of conferences *États de la Recherche* organized regularly by the Societé Mathématique de France. The purpose of these "State of research" national training sessions is to provide a panoramic view on today's mathematics, accessible to a wide audience of junior and senior mathematicians, not necessarily specialized in the given field of research.

It is our pleasure to thank all the speakers and the participants for the success of this conference, to acknowledge the financial and technical support of the Société Mathématique de France and the Institut Universitaire de France, and the hospitality of the Institut Henri Poincaré. We are grateful to Arnaud Guillin, from the editorial committee of *Panoramas et Synthèses*, who suggested to us to forge this book from the conference lectures notes. We are indebted to Walid Hachem and to Justin Salez for their help during the preparation of the draft of this book.

F. Benaych-Georges, D. Chafaï, S. Péché, B. de Tilière.

1. The conference website is `http://djalil.chafai.net/wiki/erma:start`.

CHAPTER 1

LOCAL SEMICIRCLE LAW FOR WIGNER MATRICES

by Florent Benaych-Georges and Antti Knowles.

These notes provide an introduction to the local semicircle law from random matrix theory, as well as some of its applications. We focus on *Wigner matrices*, Hermitian random matrices with independent upper-triangular entries with zero expectation and constant variance. We state and prove the local semicircle law, which says that the eigenvalue distribution of a Wigner matrix is close to Wigner's semicircle distribution, down to spectral scales containing slightly more than one eigenvalue. This local semicircle law is formulated using the Green function, whose individual entries are controlled by large deviation bounds.

We then discuss three applications of the local semicircle law: first, complete delocalization of the eigenvectors, stating that with high probability the eigenvectors are approximately flat; second, rigidity of the eigenvalues, giving large deviation bounds on the locations of the individual eigenvalues; third, a comparison argument for the local eigenvalue statistics in the bulk spectrum, showing that the local eigenvalue statistics of two Wigner matrices coincide provided the first four moments of their entries coincide. We also sketch further applications to eigenvalues near the spectral edge, and to the distribution of eigenvectors.

1.1. Introduction

These notes are based on lectures given by Antti Knowles at the conference *États de la recherche en matrices aléatoires* at the Institut Henri Poincaré in December 2014. In them, we state and prove the local semicircle law and give several applications to the distribution of eigenvalues and eigenvectors. We favor simplicity and transparency of arguments over generality of results. In particular, we focus on one of the simplest ensembles of random matrix theory: Wigner matrices.

A Wigner matrix is a Hermitian random matrix whose entries are independent up to the symmetry constraint, and have zero expectation and constant variance. This definition goes back to the seminal work of Wigner **[79]**, where he also proved the semicircle law, which states that the asymptotic eigenvalue distribution of a Wigner matrix is given with high probability by the semicircle distribution. Wigner matrices occupy a central place in random matrix theory, as a simple yet nontrivial ensemble on which many of the fundamental features of random matrices, such as universality of the local eigenvalue statistics and eigenvector delocalization, may be analyzed. Moreover, many of the techniques developed for Wigner matrices, such as the ones presented in these notes, extend to other random matrix ensembles or serve as a starting point for more sophisticated methods.

The local semicircle law is a far-reaching generalization of Wigner's original semicircle law, and constitutes the key tool for analyzing the local distribution of eigenvalues and eigenvectors of Wigner matrices, and in particular in the study of the universality of Wigner matrices. Roughly, it states that the eigenvalue distribution is well approximated by the semicircle distribution down to scales containing only slightly more than one eigenvalue. The first instance of the local semicircle law down to optimal spectral scales was obtained by Erdős, Schlein, and Yau in [**39**], following several previous results on larger spectral scales [**40, 52**]. Since then, the local semicircle law has been improved and generalized in a series of works [**41, 38, 44, 43, 45, 37, 36, 26**]. The proof presented in these notes is modeled on the argument of Erdős, Knowles, Yau, and Yin from [**36**], which builds on the works [**40, 39, 41, 44, 43, 45**] of Erdős, Schlein, Yau, and Yin.

In order to keep these notes focused, for the applications we restrict ourselves to relatively simple consequences of the semicircle law: eigenvalue rigidity, eigenvector delocalization, and a four-moment comparison theorem for the local eigenvalue statistics. Further topics and applications, such as distribution of eigenvalues near the spectral edges, Dyson Brownian motion and its local relaxation time, distribution of eigenvectors, and spectral statistics of deformed matrix ensembles, are not covered here. For further reading on Wigner matrices, we recommend the books [**60, 3**]. Additional applications of the local semicircle law, in particular in the analysis of the local relaxation time of Dyson Brownian motion, are given in the survey [**30**]. In Section 1.11.4, we briefly outline applications of the local semicircle law to the analysis of eigenvectors and eigenvalues near the spectral edge.

Outline. – In Section 1.2 we define Wigner matrices and state the local semicircle law, or *local law* for short. We also give two simple consequences of the local law: *eigenvalue rigidity* and *complete eigenvector delocalization*. Sections 1.3–1.7 are devoted to the proof of the local law. Section 1.3 collects some basic tools from linear algebra and probability theory that are used throughout the proof. Section 1.4 gives a detailed outline of the proof. In Section 1.5 we perform the first of two major steps of the proof: the weak local law, which yields control down to optimal scales but with non-optimal error bounds. In Section 1.6 we perform the second major step of the proof, which yields optimal error bounds and concludes the proof of the local law. The key estimate used to obtain the optimal error bounds is a fluctuation averaging result, whose proof is given in Section 1.7.

Having concluded the proof of the local law, in Sections 1.8–1.10 we draw some simple consequences. In Section 1.8 we prove the semicircle law on small scales, which provides large deviation bounds on the number of eigenvalues in small intervals. In Section 1.9 we prove eigenvalue rigidity, which provides large deviation bounds on the locations of individual eigenvalues. In Section 1.10 we extend the estimates from the local law to arbitrary scales and distances from the spectrum.

In Section 1.11 we illustrate how to use the local law to obtain a four-moment comparison theorem for the local eigenvalue statistics, using the Green function comparison method from [**44**]. We also sketch how to extend such comparison methods to the edge of the spectrum and to eigenvectors. Finally, in Section 1.12 we discuss further generalizations of the local law, and also other random matrix models for which local laws have been established.

The appendices contain some basic tools from linear algebra, spectral theory, and probability that are used throughout the notes, as well as some standard results on the semicircle distribution and the norm of Wigner matrices.

Pedagogical aspirations aside, in these notes we also give a coherent summary of the different guises of the local semicircle law that have proved useful in random matrix theory. They have all appeared, at least implicitly, previously in the literature; we take this opportunity to summarize them explicitly in Sections 1.2 and 1.10.

Conventions. – We use C to denote a generic large positive constant, which may depend on some fixed parameters and whose value may change from one expression to the next. Similarly, we use c to denote a generic small positive constant. For two positive quantities A_N and B_N depending on N we use the notation $A_N \asymp B_N$ to mean $C^{-1}A_N \leqslant B_N \leqslant CA_N$ for some positive constant C. In statements of assumptions, we use the special constant $\tau > 0$, which should be thought of as a fixed arbitrarily small number. A smaller τ always leads to a weaker assumption. We always use the Euclidean norm $|\mathbf{v}|$ for vectors $\mathbf{v}$, and denote by $\|A\|$ the associated operator norm of a matrix A. Finally, in some heuristic discussions we use the symbols $\ll$, $\gg$, and $\approx$ to mean "much less than", "much greater than", and "approximately equal to", respectively.

1.2. The local law

Let $H = H^* = (H_{ij} : 1 \leqslant i,j \leqslant N) \in \mathbb{C}^{N\times N}$ be an $N \times N$ random Hermitian matrix. We normalize H so that its eigenvalues are typically of order one, $\|H\| \asymp 1$. A central goal of random matrix theory is to understand the distribution of the eigenvalues $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_N$ and the normalized associated eigenvectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_N$ of H.

Since most quantities that we are interested in depend on N, we shall almost always omit the explicit argument N from our notation. Hence, every quantity that is not explicitly a constant is in fact a sequence indexed by $N \in \mathbb{N}$.