AN EXCURSION INTO p-ADIC HODGE THEORY: FROM FOUNDATIONS TO RECENT TRENDS

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***Abstract.* —** This volume offers a progressive and comprehensive introduction to p-adic Hodge theory. It starts with Tate's works on p-adic divisible groups and the cohomology of p-adic varieties, which constitutes the main concrete motivations for the development of p-adic Hodge theory. It then moves smoothly to the construction of Fontaine's p-adic period rings and their apparition in several comparison theorems between various p-adic cohomologies. Applications and generalizations of these theorems are subsequently discussed. Finally, Scholze's modern vision on p-adic Hodge theory, based on the theory of perfectoids, is presented.

***Résumé* (Une promenade dans la théorie de Hodge p-adique : des fondements aux développements récents). —** Ce volume propose une introduction progressive à la théorie de Hodge p-adique. En guise d'introduction, le lecteur est invité à découvrir les travaux de Tate sur les groupes p-divisibles et la cohomologie des variétés p-adiques qui contiennent en essence les prémisses de la théorie de Hodge p-adique. À la suite de cette initiation, la lectrice est guidée naturellement vers la définition des anneaux de Fontaine de périodes p-adiques et leur apparition dans certains théorèmes de comparaison entre diverses cohomologies p-adiques. Des applications et des généralisation de ces théorèmes sont discutées par la suite. Le volume se conclut par une exposition de la vision moderne de la théorie de Hodge p-adique, qui est dûe à Scholze et est fondée sur la notion de perfectoïdes.

CONTENTS

INTRODUCTION

Xavier Caruso

This volume is dedicated to Jean-Marc Fontaine who passed away in January 2019.

Since the introduction of algebraic methods in topology by Poincaré at the end of the 19th century, the cohomology of manifolds has been intensively studied by many authors in many different directions. Among them, an important case of interest is that of complex algebraic varieties, whose cohomology has a very rich structure. For these manifolds, we have at our disposal (at least) two differential cohomological theories: the singular cohomology, which is purely topological (it makes sense for any topological space), and the de Rham cohomology, which has a analytic flavor since it is defined using differential forms. In 1931, de Rham proved a spectacular and quite unexpected (through very classical nowadays) theorem, stating that these two cohomologies are actually the same. Precisely, whenever X is a complex smooth manifold, we have a canonical isomorphism:

$$H^r_{\mathrm{sing}}(X, \mathbb{C}) \simeq H^r_{\mathrm{dR}}(X).$$

Soon after that, Hodge observed that, when X is a projective algebraic complex variety [(1)], the decomposition of any smooth differential form as a sum of a holomorphic and an antiholomorphic ones induces a canonical splitting of the de Rham cohomology of X:

$$H^r_{\mathrm{dR}}(X) = \bigoplus_{a+b=r} H^{a,b}(X)$$

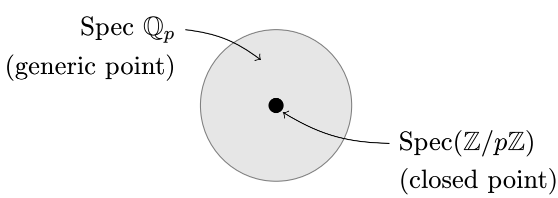
on which the complex conjugacy acts by swapping $H^{a,b}(X)$ and $H^{b,a}(X)$. Moreover, the components $H^{a,b}(X)$ have a strong geometrical interpretation in terms of Dolbeault cohomology. Throughout the 20th century, Hodge decomposition has become a fundamental tool in complex geometry. For example, viewed as a varying flag in a complex vector space (namely $H^r_{\mathrm{dR}}(X) \simeq H^r_{\mathrm{sing}}(X, \mathbb{C})$), it provides efficient methods

[(1)] Or, more generally, when X is complex, compact and carries a Kähler form.

for classifying complex algebraic varieties and, consequently, helping in solving moduli problems.

Another important breakthrough of the 20th century is the emergence of geometric approaches for attacking arithmetical questions. The notion of algebraic scheme, introduced by Grothendieck in the 1950's, leads to a uniform language in which all usual geometric constructions—including cohomology—extend to arbitrary base rings, and especially to the field of rational numbers $\mathbb{Q}$ which has, of course, a strong arithmetical taste. It is well known that the field of reals numbers $\mathbb{R}$ is obtained from $\mathbb{Q}$ by completion. It turns out that $\mathbb{Q}$ carries other absolute values, which are as least as relevant as the standard absolute value regarding arithmetics. These absolute values are indexed by the prime numbers p and are called the p-adic absolute values. By completion, they lead to the fields of p-adic numbers, denoted by $\mathbb{Q}_p$, which then appear as a natural arithmetical analog of $\mathbb{R}$. For these reasons, mathematicians start to seek for an analog of Hodge decomposition theorem in the p-adic setting.

Before going further, we would like to underline that, although $\mathbb{Q}_p$ and $\mathbb{R}$ share some similarities, they also differ in several points. Actually, the main differences are of algebraic nature. Indeed, contrary to $\mathbb{R}$, the unit ball is $\mathbb{Q}_p$ carries a structure of ring. It is called the ring of p-adic integers and is usually denoted by $\mathbb{Z}_p$. The spectrum of $\mathbb{Z}_p$ consists of two points: one *special* point, which is closed and corresponds to the unique maximal ideal $p\mathbb{Z}_p$ and one *generic* point, which is open and dense and corresponds to the ideal $\{0\}$. Spec $\mathbb{Z}_p$ is sometimes represented as follows:



On this drawing, we clearly see that the generic point, which is canonically isomorphic to Spec $\mathbb{Q}_p$, is not quite the analog of the point, but rather of the punctured unit disk. For this reason, it is sometimes relevant to consider p-adic varieties as the analogs of *families* of varieties (indexed by the "parameter" p) and it turns out that p-adic Hodge theory shares many similarities with *relative* Hodge theory whose aim is to study variations of Hodge structure by means of differential techniques (as Gauss-Manin connection).

The second difference between $\mathbb{Q}_p$ and $\mathbb{R}$ we would like to stress concerns Galois theory: whereas $\mathbb{R}$ admits only one algebraic extension, namely $\mathbb{C}$, the algebraic closure $\bar{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ has infinite degree over $\mathbb{Q}_p$ and contains many interesting subfields. In other words, while the absolute Galois group of $\mathbb{R}$ is dramatically simple (it consists of two elements: the identity and the conjugacy), the absolute Galois group of $\mathbb{Q}_p$ is much more intricated, reflecting partially the incredible richness of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. Making apparent the action of $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ is something of prime importance in p-adic Hodge theory, at which point that p-adic Hodge theory provides nowadays the most powerful tools for studying Galois representations.

In the p-adic setting, the singular cohomology is no longer relevant since basically the topology on $\mathbb{Q}_p$ is quite different from that on standard simplexes. Since Grothendieck, we know that it has to be replaced by the algebraic étale ℓ-adic cohomology where ℓ is an auxiliary prime number. This cohomology group will be denoted by $H^r_{\text{ét}}(X_{\bar{\mathbb{Q}}_p}, \mathbb{Q}_\ell)$ where X is the variery over $\mathbb{Q}_p$ we are considering. It is important to mention that $H^r_{\text{ét}}(X_{\bar{\mathbb{Q}}_p}, \mathbb{Q}_\ell)$ comes equipped with an action of $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$. De Rham cohomology admits an algebraic analog in the p-adic setting as well; we shall denote it by $H^r_{\mathrm{dR}}(X)$ in what follows. It is a vector space over $\mathbb{Q}_p$ equipped with the so-called de Rham filtration $\mathrm{Fil}^i H^r_{\mathrm{dR}}(X)$. If X were a complex variety, the filtration would be given by the formula:

$$\mathrm{Fil}^i H^r_{\mathrm{dR}}(X) = \bigoplus_{\substack{a+b=r \\ a\leq i}} H^{a,b}(X)$$

showing that it is closely related to the Hodge decomposition. However, in the p-adic case, we can prove that the de Rham filtration is not split in general.

Let us now assume that X is a projective smooth variety over $\mathbb{Q}_p$. The p-adic étale cohomology of X and its de Rham cohomology are then finite dimensional $\mathbb{Q}_p$-vector spaces with the same dimension [(2)]. Inspired by the complex case, one raises the following question—sometimes referred to as Grothendieck's *mysterious functor* problem—which can be considered as the starting point of p-adic Hodge theory.

Is there a canonical way to compare $H^r_{\text{ét}}(X_{\bar{\mathbb{Q}}_p}, \mathbb{Q}_p)$ and $H^r_{\mathrm{dR}}(X)$ (equipped with their additional structures), and to go back and forth between them?

The first significant result towards Grothendieck's question is due of Tate and appears in his seminal paper on p-divisible groups published in 1966; it states that, when A is a smooth abelian scheme over Spec $\mathbb{Z}_p$, we have a $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$-equivariant isomorphism:

$$(1) \qquad \mathbb{C}_p \otimes_{\mathbb{Q}_p} H^1_{\text{ét}}(A_{\bar{\mathbb{Q}}_p}, \mathbb{Q}_p) \simeq \big(\mathbb{C}_p \otimes_K H^1(A, \mathcal{O}_A)\big) \oplus \big(\mathbb{C}_p(-1) \otimes_K H^0(A, \Omega_{A/K})\big)$$

where $\mathbb{C}_p$ denotes the completion of $\bar{\mathbb{Q}}_p$ and $\mathbb{C}_p(-1)$ is its twist by the inverse of the cyclotomic character. The right hand side of (1) is not quite the de Rham cohomology of A, but is nevertheless related to it since it is isomorphic to its graded module with respect to the de Rham filtration.

After this result, Grothendieck's question has been investigated by Fontaine for several decades. After several partial results (including an extension of Tate's theorem to all abelian varieties over $\mathbb{Q}_p$), Fontaine managed, in the 1990's, to introduce the ingredients that will eventually allow for a complete answer to Grothendieck's question. Precisely, Fontaine constructed a large field B_{dR}, the so-called field of p-adic periods and, together with Jannsen, he formulated the (C_{dR})-conjecture, stating that there should exist a canonical isomorphism (compatible with all additional structures)

(2) This can be proved by reduction to the complex case after the choice of a field embedding $\mathbb{Q}_p \hookrightarrow \mathbb{C}$.

between the étale cohomology and the de Rham cohomology after extending scalars to B_{dR}, i.e.,

$$B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} H^r_{\text{ét}}(X_{\bar{\mathbb{Q}}_p}, \mathbb{Q}_p) \simeq B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} H^r_{\mathrm{dR}}(X) \tag{2}$$

for all nonnegative integer r and all proper smooth variety X over $\mathbb{Q}_p$. Let us underline that this isomorphism could be thought of as a p-adic analog of the classical Hodge decomposition theorem since, beyond relating two different types of cohomologies, it shows that the cohomology of X is endowed with remarkable additional structures, which are a filtration and a Galois action preserving it.

Besides, Fontaine noticed that the situation should be even richer when X admits a nice prolongation $\mathscr{X}$ to Spec $\mathbb{Z}_p$ (that is, following our analogy, to the "unpunctured" unit disk), i.e., when we assume that the degeneracy of X at the special point of Spec $\mathbb{Z}_p$ remains under control. The simplest situation occurs when the prolongation $\mathscr{X}$ remains smooth, i.e., when there is no degeneracy. This is the so-called case of *good reduction*. Another case of interest occurs when the special fiber of $\mathscr{X}$ is a normal crossing divisor. This is the so-called case of *semi-stable reduction*. In both cases, one can relate the de Rham cohomology of X with a suitable cohomology of the special fiber of $\mathscr{X}$. Since the latter is defined over $(\mathbb{Z}/p\mathbb{Z})$, one derives a Frobenius action on $H^r_{\mathrm{dR}}(X)$. Moreover, in the case of semi-stable reduction, $H^r_{\mathrm{dR}}(X)$ comes also equipped with a monodromy action reflecting, roughly speaking, how the cohomology is changed when one turns around the singularity. Fontaine suggested that, in the case of good reduction (resp. semi-stable reduction), the isomorphism (2) could be strengthened and incorporate all the additional structures we have at our disposal. Precisely, Fontaine defined two subrings B_{crys} and B_{st} of B_{dR} and conjectured that:

$$\begin{aligned} \text{when } X \text{ has good reduction:} \quad & B_{\mathrm{crys}} \otimes_{\mathbb{Q}_p} H^r_{\text{ét}}(X_{\bar{\mathbb{Q}}_p}, \mathbb{Q}_p) \simeq B_{\mathrm{crys}} \otimes_{\mathbb{Q}_p} H^r_{\mathrm{dR}}(X) \\ \text{when } X \text{ has semi-stable reduction:} \quad & B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} H^r_{\text{ét}}(X_{\bar{\mathbb{Q}}_p}, \mathbb{Q}_p) \simeq B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} H^r_{\mathrm{dR}}(X) \end{aligned}$$

these isomorphisms being compatible with all structures: Galois action, filtration, Frobenius action and monodromy action in the semi-stable case. Fontaine also observed that, using all these additional structures, the refined isomorphisms above could be used to go back and forth between $H^r_{\text{ét}}(X_{\bar{\mathbb{Q}}_p}, \mathbb{Q}_p)$ and $H^r_{\mathrm{dR}}(X)$, providing then a complete satisfying answer to Grothendieck's mysterious functor problem.

After consequent works of many authors, Fontaine's conjectures have been proved, and important consequences have been derived. Typically, they allow for a powerful description of the Galois action on the étale cohomology, from which interesting information can be derived (as description of Serre's weights, ramification bounds, *etc.*) A beautiful illustration of the type of results one may reach with these techniques is a theorem of Fontaine asserting that there is no smooth abelian scheme over Spec $\mathbb{Z}$. Nowadays, Fontaine's developments have become crucial for many applications in Arithmetic Geometry and Number Theory, including deformation spaces of Galois representation, modularity lifting theorems, Langlands correspondence, *etc.*