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CENTRAL POINTS OF THE DOUBLE HEPTAGON TRANSLATION SURFACE ARE NOT CONNECTION POINTS

BY JULIEN BOULANGER

ABSTRACT. — We consider flow directions on the translation surfaces formed from double $(2n+1)$-gons and give a sufficient condition in terms of a natural continued fractions algorithm for a direction to be hyperbolic in the sense that it is a fixed direction for some hyperbolic element of the Veech group of the surface. In particular, we give explicit points with coordinates in the trace field of the double heptagon translation surface, that are not so-called connection points. Among these are the central points of the heptagons, giving a negative answer to a question by P. Hubert and T. Schmidt [1].

RÉSUMÉ (*Les points centraux du double heptagone ne sont pas des points de connexion*). — On s'intéresse au flot directionnel sur les surfaces de translation obtenues à partir de deux $(2n+1)$-gones dont on a recollé les côtés parallèles, et on donne une condition suffisante pour qu'une direction soit hyperbolique, c'est à dire fixée par une direction hyperbolique du groupe de Veech, en termes d'un algorithme de fractions continues naturel sur les directions de la surface. En particulier, cela nous permet d'exhiber des points sur le double heptagone à coordonnées dans le corps de trace qui ne sont pas des points de connexion. Parmi ces points on peut notamment trouver les points centraux des heptagones, ce qui donne une réponse négative à une question de P. Hubert et T. Schmidt [1].

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1. Introduction and statement of the results

A translation surface is a genus g topological surface with an atlas of charts on the surface minus a finite set of points such that all transition functions are translations. These surfaces can also be described as the surfaces obtained by gluing pairs of opposite parallel sides of a collection of Euclidean polygons by translations. Such surfaces arise naturally in the study of billiard table dynamics: the Katok–Zemlyakov unfolding procedure, which consists in reflecting the billiard every time the trajectory hits an edge instead of reflecting the trajectory, replaces the billiard flow on a polygon by a directional flow on isometric translation surfaces. The study of translation surfaces has been flourishing, with major recent advances such as the results in [12], [10], or [11], but there still remains various open questions, for instance in the area of Veech groups. One of these questions is to characterize so-called connection points, for which little is known for translation surfaces whose trace field is of degree 3 or more over $\mathbb{Q}$. In this paper, we look at two particular points of the double heptagon surface, whose trace field is cubic over $\mathbb{Q}$, and show that they are not connection points. For surveys about translation surfaces, see [25] and [24], and for Veech groups, see [16].

Before looking at connection points, one needs to understand better parabolic (or hyperbolic) directions; that is, directions fixed by a parabolic (or hyperbolic) element of the Veech group. For Veech surfaces, periodic directions, saddle connection directions and directions fixed by parabolic elements of the Veech group coincide. For these terms, see the background and [16]. For translation surfaces whose trace field is quadratic or $\mathbb{Q}$, C. McMullen showed in [18] that (after a natural normalization) the periodic directions are exactly those with slopes in the trace field. When the trace field is of higher degree, it is no longer true, and the periodic directions in general form a proper subset of the directions whose slope belong to the trace field. D. Davis and S. Lelièvre [8] characterized the parabolic directions for the double pentagon surface using a continued fractions algorithm. Their results can be directly extended to the $(2n+1)$-gon, which has a trace field of degree n over $\mathbb{Q}$.

In this paper, we use the algorithm to characterize hyperbolic directions whose slopes belong to the trace field for each double $(2n+1)$-gon surface, which are made of two copies of a $(2n+1)$-gon with parallel opposite sides glued together. We find explicit examples of such directions for the double heptagon. This allows us to prove that central points of the double heptagon are not connection points, see Theorem 1.3. This answers negatively a question of P. Hubert and T. Schmidt. Recall that the central points of the double heptagon are the centers of the heptagons. A nonsingular point of a translation surface is called a connection point if every separatrix passing through this point can be extended to a saddle connection. In fact, the author does not know any example of a nonperiodic connection point[1] for a translation surface whose trace field is of degree 3 over $\mathbb{Q}$ or higher.

THEOREM 1.1. — *Let $n \geq 2$, for the double $(2n+1)$-gon surface, the directions that end in a periodic sequence (of period ≥ 2) for the continued fractions algorithm are hyperbolic directions.*

PROPOSITION 1.2 (Double heptagon case). — *For the double heptagon surface, there are hyperbolic directions in the trace field.*

This proposition is already known from [2] and [13], where a different method is used. Our method provides an answer to the question of central points as connection points, which was not known.

THEOREM 1.3. — *Central points of the double heptagon are not connection points.*

Moreover, one can look at double $(2n+1)$-gons with more sides. For example, the same result holds for the double nonagon:

THEOREM 1.4. — *Central points of the double nonagon are not connection points.*

Moreover, various tests that we conducted suggest the following conjecture, which is not new since we found the same ideas in [13].

CONJECTURE 1.5. — *For the double heptagon and the double nonagon, all the directions in the trace field are either parabolic or hyperbolic.*

What is interesting is that these results do not seem to generalize to the double hendecagon, for example. In fact, for the double hendecagon, we were not able to find any direction in the trace field that ends in a periodic sequence. These issues will be discussed in Section 5.

2. Background

A *translation surface* (X, ω) is a real compact genus g surface X with an atlas ω such that all transition functions are translations except on a finite set of singularities Σ, along with a distinguished direction. Alternatively, it can be seen as a surface obtained from a finite collection of polygons embedded in $\mathbb{C}$ by gluing pairs of parallel opposite sides by translation. We get a surface X with a flat metric and a finite number of singularities. We define $X' = X - \Sigma$, which inherits the translation structure of X and defines a Riemannian structure on X'. Therefore, we have notions of geodesics, length, angle, and geodesic

1. A point is *periodic* if its orbit under the action of the affine group is finite, otherwise it is nonperiodic, see [15].