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ON THE MONODROMY MAP FOR LOGARITHMIC DIFFERENTIAL SYSTEMS

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ABSTRACT. — We study the monodromy map for logarithmic $\mathfrak{g}$-differential systems over an oriented surface S_0 of genus g, with $\mathfrak{g}$ being the Lie algebra of a complex reductive affine algebraic group G. These logarithmic $\mathfrak{g}$-differential systems are triples of the form (X, D, Φ), where $(X, D) \in \mathcal{T}_{g,d}$ is an element of the Teichmüller space of complex structures on S_0 with $d \geq 1$ ordered marked points $D \subset S_0 = X$ and Φ is a logarithmic connection on the trivial holomorphic principal G-bundle $X \times G$ over X, whose polar part is contained in the divisor D. We prove that the monodromy map from the space of logarithmic $\mathfrak{g}$-differential systems to the character variety of

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G-representations of the fundamental group of $S_0 \setminus D$ is an immersion at the generic point in the following two cases:

1. $g \geq 2$, $d \geq 1$, and $\dim_{\mathbb{C}} G \geq d+2$;
2. $g = 1$ and $\dim_{\mathbb{C}} G \geq d$.

The above monodromy map is nowhere an immersion in the following two cases:

1. $g = 0$ and $d \geq 4$;
2. $g \geq 1$ and $\dim_{\mathbb{C}} G < \frac{d+3g-3}{g}$.

This extends to the logarithmic case the main results in [5], [2] dealing with nonsingular holomorphic $\mathfrak{g}$-differential systems (which corresponds to the case of $d = 0$).

Résumé (*Sur la monodromie des systèmes différentiels logarithmiques*). — Nous étudions la monodromie des $\mathfrak{g}$-systèmes différentiels logarithmiques au-dessus d'une surface compacte orientée S_0 de genre g, où $\mathfrak{g}$ désigne l'algèbre de Lie d'un groupe de Lie complexe affine réductif G. Ces $\mathfrak{g}$-systèmes différentiels sont des triplets de la forme (X, D, Φ), où $(X, D) \in \mathcal{T}_{g,d}$ est un élément de l'espace de Teichmüller de structures complexes sur S_0, avec $d \geq 1$ points marqués ordonnés $D \subset S_0 = X$ et Φ est une connexion logarithmique sur le G-fibré holomorphe trivial $X \times G$ au-dessus de X et dont la partie polaire est contenue dans le diviseur D.

Nous démontrons que l'application de monodromie définie sur l'espace des $\mathfrak{g}$-systèmes différentiels logarithmiques et à valeurs dans la variété des caractères de G-représentations du groupe fondamental de $S_0 \setminus D$ est une immersion au point générique dans les deux cas suivants :

1. $g \geq 2$, $d \geq 1$, et $\dim_{\mathbb{C}} G \geq d+2$;
2. $g = 1$ et $\dim_{\mathbb{C}} G \geq d$.

L'application de monodromie ci-dessus n'est en aucun point une immersion dans les deux cas suivants :

1. $g = 0$ et $d \geq 4$;
2. $g \geq 1$ et $\dim_{\mathbb{C}} G < \frac{d+3g-3}{g}$.

Ceci étend au cas logarithmique les résultats principaux de [5], [2] qui traitent le cas des $\mathfrak{g}$-systèmes différentiels holomorphes non singuliers (qui correspondent ici au cas $d = 0$).

1. Introduction

The study of the Riemann–Hilbert mapping, which associates to a flat (algebraic or holomorphic) connection its monodromy morphism from the fundamental group is a classical topic in algebraic and analytical geometry (see, for instance, [8], [17], and references therein).

We recall the setup and results of [5] and [2], the predecessors of this paper. Let G be a connected reductive affine algebraic group defined over $\mathbb{C}$, with $\dim G > 0$, and let $\mathfrak{g}$ be the Lie algebra of G. A $\mathfrak{g}$-differential system is a pair of the form (X, Φ), where X is a complex structure on a compact oriented smooth surface S_0 of genus g, and Φ is a holomorphic connection on the trivial holomorphic principal G-bundle $X \times G$ over the Riemann surface X. A $\mathfrak{g}$-differential system (X, Φ) is called irreducible if Φ is not induced by a holomorphic connection on $X \times P$ for some proper parabolic subgroup P of G.

Since any holomorphic connection on a Riemann surface is flat, associating the monodromy representation to a holomorphic connection we obtain a map from the space of irreducible $\mathfrak{g}$-differential systems to the irreducible G-character variety $\mathrm{Hom}(\pi_1(S_0), G)^{\mathrm{ir}}/G$. This monodromy map is actually holomorphic.

The main result of [5] says that, if $g = 2$, this Riemann–Hilbert monodromy map is a local diffeomorphism from the space of irreducible $\mathfrak{g}$-differential systems into the irreducible G-character variety, for $G = \mathrm{SL}(2, \mathbb{C})$. Being inspired by [5], in [2] it was shown that, for all $g \geq 2$, the above monodromy map is an immersion on an open dense subset of the space of irreducible $\mathfrak{g}$-differential systems, for all reductive groups G with $\dim_{\mathbb{C}} G \geq 3$.

Our aim here is to study the Riemann–Hilbert monodromy mapping for logarithmic $\mathfrak{g}$-differential systems, where $\mathfrak{g}$ is as above. These logarithmic $\mathfrak{g}$-differential systems are defined by triples of the form (X, D, Φ), where $(X, D) \in \mathcal{T}_{g,d}$ is an element of the Teichmüller space of complex structures on S_0, with d ordered marked points $D \subset S_0 = X$ (see Section 3), and Φ is a logarithmic connection on the trivial holomorphic principal G-bundle $X \times G$ over X, whose polar part is contained in the divisor D.

We prove the following (see Theorem 4.4):

Theorem 1.1. — *Assume that $3g - 3 + d > 0$ and $d \geq 1$. The Riemann–Hilbert monodromy mapping from the above space of irreducible logarithmic $\mathfrak{g}$-differential systems to the character variety of irreducible G-representations of the fundamental group of $S_0 \setminus D$ is an immersion at the generic point in the following two cases:*

1. $g \geq 2$ *and* $\dim_{\mathbb{C}} G \geq d + 2$*;*
2. $g = 1$ *and* $\dim_{\mathbb{C}} G \geq d$*.*

The Riemann–Hilbert monodromy mapping from the above space of irreducible logarithmic $\mathfrak{g}$-differential systems to the character variety of irreducible G-representations of the fundamental group of $S_0 \setminus D$ is nowhere an immersion in the following two cases:

1. $g = 0$*;*
2. $g \geq 1$ *and* $\dim_{\mathbb{C}} G < \frac{d+3g-3}{g}$ *(in particular, when $g = 1$ and $\dim_{\mathbb{C}} G < d$).*

We note that Theorem 1.1 gives a complete answer only when $g = 0$ or $g = 1$. For given $g \geq 2$ and G, there are finitely many cases of d that are not addressed in Theorem 1.1. When $g = 1$ and $d = 0$, from the first part of Theorem 1.1 it follows that the monodromy mapping from the space of irreducible logarithmic $\mathfrak{g}$-differential systems is an immersion at the generic point; see Remark 4.6.

Theorem 1.1, extends to the class of logarithmic $\mathfrak{g}$-differential systems, the main result in [2] which deals with the nonsingular holomorphic $\mathfrak{g}$-differential systems (corresponding to the case $d = 0$). Notice that the hypothesis $3g - 3 + d > 0$ in Theorem 1.1 implies that the above Teichmüller space $\mathcal{T}_{g,d}$ has positive dimension.