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THE CLASSIFICATION OF BRANCHED WILLMORE SPHERES IN THE 3-SPHERE AND THE 4-SPHERE

BY Alexis MICHELAT AND Tristan RIVIÈRE

Abstract. – We extend the classification of Robert Bryant of Willmore spheres in S^3 to *variational* branched Willmore spheres S^3 and show that they are inverse stereographic projections of complete minimal surfaces with finite total curvature in $\mathbb{R}^3$ and vanishing flux. We also obtain a classification of *variational* branched Willmore spheres in S^4, generalizing a theorem of Sebástian Montiel. The other main result of the article is a new local criterion which implies that branched Willmore spheres are conformally minimal.

Résumé. – Nous étendons la classification de Robert Bryant des sphères de Willmore de S^3 aux sphères de Willmore branchées et *variationelles* de S^3 et montrons que ces dernières sont des projections stéréographiques inverses de surfaces minimales de $\mathbb{R}^3$ ayant une courbure totale finie et un flux nul. Nous obtenons également une classification des sphères de Willmore branchées et *variationelles* de S^4, généralisant ainsi un théorème de Sebástian Montiel. L'autre théorème principal de l'article est un critère local impliquant qu'une sphère de Willmore branchée est conformément minimale.

1. Introduction

1.1. Willmore functional and quantization of energy

This article primarily addresses the generalization of Bryant's classification of smooth Willmore immersions of the sphere S^2 into S^3 to branched immersions. Let Σ be a closed Riemann surface, and $n \geq 3$ be a fixed integer. The Willmore energy on a smooth Riemannian manifold (M^n, h) with sectional curvature $\widetilde{K}_h$ is defined on any smooth immersion $\vec{\Phi} : \Sigma \to M^n$ by

$$W_{M^n}(\vec{\Phi}) = \int_\Sigma \left(|\vec{H}_g|^2 + \widetilde{K}_h\right) d\,\mathrm{vol}_g,$$

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where $g = \vec{\Phi}^* h$ is the pull-back metric of (M^n, h) by $\vec{\Phi}$, and $\vec{H}_{\vec{\Phi}}$ is the mean-curvature, that is the half-trace of the second fundamental form $\vec{\mathbb{I}}$ the immersion, given by

$$\vec{H}_g = \frac{1}{2} \sum_{i,j=1}^{2} g^{i,j}\, \vec{\mathbb{I}}_{i,j}.$$

The most basic property of the Willmore energy is its conformal invariance which can be stated as follows. For all conformal diffeomorphism $\varphi : (M^n, h) \to (\widetilde{M}^n, \widetilde{h})$, we have

$$W_{\widetilde{M}^n}(\varphi \circ \vec{\Phi}) = W_{M^n}(\vec{\Phi}).$$

However, in the special case of $\mathbb{R}^n$, if $\iota_a : \mathbb{R}^n \setminus \{a\} \to \mathbb{R}^n \setminus \{a\}$ is the inversion centered at $a \in \vec{\Phi}(\Sigma)$, we do not have in general

$$W_{\mathbb{R}^n}(\iota_a \circ \vec{\Phi}) = W_{\mathbb{R}^n}(\vec{\Phi}),$$

while we have the equality for inversions with center outside of $\vec{\Phi}(\Sigma)$. Nevertheless, the quantity

$$\mathscr{W}(\vec{\Phi}) = \int_\Sigma \left(|\vec{H}_g|^2 - K_g \right) d\mathrm{vol}_g,$$

where K_g is the intrinsic Gauss curvature of $\vec{\Phi}$, is invariant under every conformal transformation. Indeed, the 2-form

$$\left(|\vec{H}_g|^2 - K_g \right) d\mathrm{vol}_g = |\vec{h}_0|^2_{WP}\, d\mathrm{vol}_g,$$

where $\vec{h}_0$ is the Weingarten tensor and $|\cdot|_{WP}$ designs the Weil-Petersson metric, is a *pointwise* invariant (see for example 7.3.1 [35]). We shall come back to this point when we will state Noether's theorem for the Willmore energy (see for Example (89) in the proof of Theorem 3.8).

We now come to the critical points of the Willmore energy. Classically, they are the smooth immersions satisfying to the equation

$$\Delta_g^N \vec{H} - 2|\vec{H}|^2 \vec{H} + \mathscr{A}(\vec{H}) + \mathscr{R}(\vec{H}) = 0, \tag{1}$$

where Δ_g^N is the Laplacian on the normal bundle, $\mathscr{A}$ the Simons operator and $\mathscr{R}$ a curvature operator, given by

$$\mathscr{A}(\vec{H}) = \sum_{i,j=1}^{2} \langle \vec{\mathbb{I}}(\vec{\varepsilon}_i, \vec{\varepsilon}_j), \vec{H} \rangle \vec{\mathbb{I}}(\vec{\varepsilon}_i, \vec{\varepsilon}_j), \qquad \mathscr{R}(\vec{H}) = \left(\sum_{i=1}^{2} R(\vec{H}, \vec{\varepsilon}_i)\vec{\varepsilon}_i \right)^N,$$

where $(\vec{\varepsilon}_1, \vec{\varepsilon}_2)$ is any local orthonormal moving frame, and R is the Riemann curvature tensor of (M^n, h). However, for the natural regularity $\vec{\Phi} \in W^{2,2}(\Sigma, M^n)$ this equation does not even have a distributional meaning, as it would require $\vec{H} \in L^3_{\mathrm{loc}}(\Sigma, TM^n)$. The weakest possible setting to work with is the space of *weak immersions* (introduced in [30], [31])

$$\mathscr{E}(\Sigma, M^n) = W^{2,2} \cap W^{1,\infty}(\Sigma, M^n) \cap \left\{ \begin{aligned} &\vec{\Phi} : d\vec{\Phi}(x) \text{ has rank 2 for almost all } x \in \Sigma \\ &\quad \text{and } \inf_\Sigma |d\vec{\Phi} \wedge d\vec{\Phi}|_{g_0} > 0 \end{aligned} \right\}$$

for any fixed Riemannian metric g_0 on Σ. In the rest of the introduction, we suppose that $M^n = \mathbb{R}^n$ and that h is the standard flat Euclidean metric. The second author showed in the Willmore equation can be written in a conservative weak formulation.

THEOREM ([30], Theorem I.1 p. 4). – *Let Σ be a closed Riemann surface, and $\vec{\Phi} : \Sigma \to \mathbb{R}^n$ be a smooth immersion. Then, (identifying 2-vectors and functions on Σ)*

$$\Delta_g^N \vec{H}_g - 2|\vec{H}_g|^2 \vec{H}_g + \mathscr{A}(\vec{H}_g) = d\left(*_g d\vec{H}_g - 3 *_g (d\vec{H}_g)^N + \star(\vec{H}_g \wedge d\vec{n})\right), \tag{2}$$

where $\vec{H}_g$ is the mean curvature of $\vec{\Phi}$, where $*_g$ is the Hodge star operator on Σ for the metric g, and $\star$ the Hodge star operator on $\mathbb{R}^n$ for the flat metric.

As the 1-form under the exterior derivative in (2) is in $W^{-1,2} + L^1$, the right-hand side is well-defined in a distributional sense as an element of $\mathscr{D}'(\Sigma)$. If the left-hand side is not defined in general for $\vec{\Phi} \in \mathscr{E}(\Sigma, \mathbb{R}^n)$, this comes from the fact that to write it, one has to make a projection on the normal bundle, while the normal is only in $W^{1,2}(\Sigma, \mathscr{G}_{n-2}(\mathbb{R}^n))$, where $\mathscr{G}_{n-2}(\mathbb{R}^n)$ denotes the oriented Grassmannian of $(n-2)$-plans in $\mathbb{R}^n$. Computing the Euler-Lagrange equation for arbitrary variations allows one to recover the weak formulation of the right-hand side (see [24]). Furthermore, as we shall see, the conservative form of the Euler-Lagrange equation of Willmore energy is a consequence of Noether's theorem (this last fact already appears in a paper by Yann Bernard ([1] Theorem I.2 p. 220)).

Furthermore, writing the Willmore equation as the closeness of a 1-form allows one to introduce the concept of *variational* Willmore surfaces. In general, a critical point of W is smooth outside of a finite number of points (called branch points, where $\vec{\Phi}$ fails to be an immersion), but globally only in $W^{2,p}(\Sigma, \mathbb{R}^n)$ for all $p < \infty$. In particular, if the branch points are $p_1, \dots, p_m \in \Sigma$, we could have

$$d\left(*_g d\vec{H}_g - 3 *_g (d\vec{H}_g)^N + \star(\vec{H}_g \wedge d\vec{n})\right) = \sum_{i=1}^m \vec{\alpha}_i \delta_{p_i} \tag{3}$$

for some $\vec{\alpha}_1, \dots, \vec{\alpha}_m \in \mathbb{R}^n$, or more generally derivatives of Dirac masses.

DEFINITION 1. – We say that a branched Willmore immersion is *variational* if it is obtained as a weak limit or as a bubble of a sequence of Willmore immersions of uniformly bounded energy.

The Equation (3) permits to introduce the first residue defined in [2] (see formula (1-8) p. 260) as

$$\widetilde{\vec{\gamma}_0}(p_i) = \frac{1}{4\pi}\int_\gamma *_g d\vec{H}_g - 3 *_g (d\vec{H}_g)^N + \star(\vec{H}_g \wedge d\vec{n}) = \frac{1}{2}\vec{\alpha}_i \tag{4}$$

for any smooth closed curved γ around p_i, for $i = 1, \dots, m$. This quantity was first defined for immersions in codimension 1 by Kuwert and Schätzle in [17] (Lemma 4.1 p. 338), and in any codimension in [2]. We will see that $\widetilde{\vec{\gamma}_0}(p_i)$ measures on the basic first obstruction to the regularity of Willmore surfaces through the branch points.

DEFINITION 2. – *We say that a branched Willmore surface $\vec{\Phi} : \Sigma \to \mathbb{R}^n$ is a* true *branched immersion if for all branch point $p \in \Sigma$, the first residue $\widetilde{\vec{\gamma}_0}(p)$ vanishes* (i.e., $\widetilde{\vec{\gamma}_0}(p) = 0$).