ISSN 0012-9593

*quatrième série - tome 55* *fascicule 5* *septembre-octobre 2022*

# ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

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*Conformal metrics with prescribed gaussian and geodesic curvatures*

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

# Annales Scientifiques de l'École Normale Supérieure

Publiées avec le concours du Centre National de la Recherche Scientifique



**Rédaction / *Editor***

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45, rue d'Ulm, 75230 Paris Cedex 05, France.
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**Édition et abonnements / *Publication and subscriptions***

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**Tarifs**

Abonnement électronique : 441 euros.
Abonnement avec supplément papier :
Europe : 619 €. Hors Europe : 698 € ($ 985). Vente au numéro : 77 €.



ISSN 0012-9593 (print) 1873-2151 (electronic)

Directeur de la publication : Fabien Durand
Périodicité : 6 nos / an

*Ann. Scient. Éc. Norm. Sup.*
4ᵉ série, t. 55, 2022, p. 1289 à 1328

# CONFORMAL METRICS WITH PRESCRIBED GAUSSIAN AND GEODESIC CURVATURES

BY Rafael LÓPEZ-SORIANO, Andrea MALCHIODI YYY AND David RUIZ

Abstract. – We consider the problem of prescribing the Gaussian and the geodesic curvatures of a compact surface with boundary by a conformal deformation of the metric. We derive some existence results using a variational approach, either by minimization of the Euler-Lagrange energy or via min-max methods. One of the main tools in our approach is a blow-up analysis of solutions, which in the present setting can have diverging volume even with uniform bounds on their Morse index. To our knowledge, this is the first time such an aspect is treated. Key ingredients in our arguments are: a blow-up analysis around a sequence of points different from local maxima; the use of holomorphic domain-variations; and Morse-index estimates.

Résumé. – Nous considérons le problème de prescrire les courbures gaussienne et géodésique d'une surface compacte à bord par une déformation conforme de la métrique. Nous obtenons des résultats d'existence en utilisant une approche variationnelle, soit en minimisant l'énergie d'Euler-Lagrange ou par les méthodes min-max. L'un des principaux outils de notre approche est une analyse d'explosion des solutions qui, dans notre cadre, peuvent avoir un volume divergent. À notre connaissance, c'est la première fois qu'un tel aspect est traité. Les ingrédients-clés de nos arguments sont: une analyse d'explosion autour d'une séquence de points différents des maxima locaux; l'utilisation de variations de domaine holomorphes; et des estimations à indice de Morse.

## 1. Introduction

A classical problem in Geometry is the prescription of the Gaussian curvature on a compact Riemannian surface $\Sigma$ under a conformal change of the metric, dating back to

R. L.-S. and D. R. have been supported by the FEDER-MINECO Grants MTM2015-68210-P, PGC2018-096422-B-100 and by J. Andalucia (FQM-116). A.M. has been supported by the project *Geometric Variational Problems* and *Finanziamento a supporto della ricerca di base* from Scuola Normale Superiore and by MIUR Bando PRIN 2015 2015KB9WPT$_{001}$. He is also member of GNAMPA as part of INdAM.

 doi:10.24033/asens.2516

[4, 34]. Denote by $\tilde{g}$ the original metric, by $g$ the conformal one, and by $e^u$ the conformal factor (that is, $g = e^u \tilde{g}$). The curvature then transforms according to the law:

$$-\Delta u + 2\tilde{K}(x) = 2K(x)e^u,$$

where $\Delta = \Delta_{\tilde{g}}$ stands for the Laplace-Beltrami operator associated to the metric $\tilde{g}$, and $\tilde{K}$, $K$ stand for the Gaussian curvatures with respect to $\tilde{g}$ and $g$, respectively. The solvability of this equation has been studied for several decades, and it is not possible to give here a comprehensive list of references. We refer the interested reader to Chapter 6 in the book [2].

If $\Sigma$ has a boundary, other than the Gaussian curvature in $\Sigma$ it is natural to prescribe also the geodesic curvature on $\partial\Sigma$. Denoting by $\tilde{h}$ and $h$ the geodesic curvatures of the boundary with respect to $\tilde{g}$ and $g$ respectively, we are led to the boundary value problem:

$$\begin{cases} -\Delta u + 2\tilde{K}(x) = 2K(x)e^u, & \text{in } \Sigma, \\ \dfrac{\partial u}{\partial n} + 2\tilde{h}(x) = 2h(x)e^{u/2}, & \text{on } \partial\Sigma. \end{cases} \tag{1.1}$$

In the literature there are results on some versions of the latter problem. For example, the case $h = 0$ has been treated in [9], while the case $K = 0$ has been considered in [7, 39, 42]. There is also work on the blow-up analysis of solutions, see [3, 13], although the phenomenon is not fully understood.

The case of constant $K$, $h$ has also been considered: in [5] the author used a parabolic flow to obtain solutions in the limit. Using methods from complex analysis and the structure of Liouville equations, explicit expressions for the solutions and the exact values of the constants were determined if $\Sigma$ is a disk or an annulus, see [30, 32]. Some classification results for the half-plane are also available in [25, 38, 50]. However, there are almost no results for the general situation in which both curvatures are variable functions: the following ones are the only ones we are aware of at the moment. In [11] partial existence results are given, but some of them include an undetermined Lagrange multiplier. A Kazdan-Warner obstruction to existence has been found in [27]. Very recently, a result for positive symmetric curvatures in the disk has appeared, see [12].

In higher dimensions the natural analogous question regards the simultaneous prescription of the scalar curvature and the boundary mean curvature. The scalar-flat case with constant mean curvature is known as the *Escobar problem*, in strong relation with the Yamabe problem. In this regard, see [1, 15, 18, 19, 23, 28, 29, 43] and the references therein.

Integrating (1.1) and applying the Gauss-Bonnet theorem, one obtains

$$\int_\Sigma K e^u + \oint_{\partial\Sigma} h e^{u/2} = 2\pi\chi(\Sigma), \tag{1.2}$$

where $\chi(\Sigma)$ is the Euler characteristic of $\Sigma$.

In this paper we study the existence and compactness of solutions of (1.1) in the negatively-curved case, namely when $K(x) < 0$. For existence, we focus on the case $\chi(\Sigma) \leq 0$. The case of the disk is intrinsically more complicated due to the non-compact effect of the group of Möbius maps, and it will be studied in a future work. However one of the main goals of the paper is the blow-up analysis given in Theorem 1.4, which is very general and applies to any PDE in the form (1.1).

It is easy to see that, via a conformal change of metric, we can always prescribe the values $h = 0$, $K = \operatorname{sgn}\chi(\Sigma)$, see Proposition 3.1. Hence, without loss of generality, we can assume that our initial metric is such that $\tilde{K}$ is constant and $\tilde{h} = 0$. The problem to study then becomes:

$$\begin{cases} -\Delta u + 2\tilde{K} = 2K(x)e^u, & \text{in } \Sigma, \\ \dfrac{\partial u}{\partial n} = 2h(x)e^{u/2}, & \text{on } \partial\Sigma, \end{cases} \tag{1.3}$$

where $\tilde{K} = \operatorname{sgn}\chi(\Sigma)$.

Problem (1.3) is the Euler-Lagrange equation of the energy functional $I : H^1(\Sigma) \to \mathbb{R}$,

$$I(u) = \int_\Sigma \left(\frac{1}{2}|\nabla u|^2 + 2\tilde{K}u - 2K(x)e^u\right) - 4\oint_{\partial\Sigma} h(x)e^{u/2}. \tag{1.4}$$

Observe that if $K < 0$, the area and boundary terms are in competition, and a priori it is not clear whether $I$ is bounded from below or not. For the statement of our results it will be convenient to define the function $\mathfrak{D} : \partial\Sigma \to \mathbb{R}$ as

$$\mathfrak{D}(x) = \frac{h(x)}{\sqrt{|K(x)|}}. \tag{1.5}$$

Notice that $\mathfrak{D}$ is scale-invariant. As we shall see in Lemma 3.2 and 3.4, the function $\mathfrak{D}$ plays a crucial role in the global behavior of the functional $I$, as well as in the blow-up analysis of solutions to (1.1), stated in Theorem 1.4.

Our first existence result deals with the case $\chi(\Sigma) < 0$.

THEOREM 1.1. – *Assume that $\tilde{K} < 0$. Let $K$, $h$ be continuous functions such that $K < 0$ and $\mathfrak{D}(p) < 1$ for all $p \in \partial\Sigma$. Then $I$ attains its infimum and hence problem* (1.3) *admits a solution. If moreover $h \leq 0$, then the solution is unique.*

The next theorems address the case $\chi(\Sigma) = 0$.

THEOREM 1.2. – *Assume that $\tilde{K} = 0$. Let $K$, $h$ be continuous functions such that $K < 0$ and*

1. $\mathfrak{D}(p) < 1$ *for all* $p \in \partial\Sigma$;
2. $\oint_{\partial\Sigma} h(x) > 0$.

*Then $I$ attains its infimum and hence problem* (1.3) *admits a solution.*

Compared to Theorem 1.2, our next result is concerned with the reversed case of the inequality for $\mathfrak{D}$ and $\oint_{\partial\Sigma} h(x)$.

THEOREM 1.3. – *Assume that $\tilde{K} = 0$. Let $K$, $h$ be $C^1$ functions such that $K < 0$ and*

1. $\mathfrak{D}(p) > 1$ *for some* $p \in \partial\Sigma$;
2. $\oint_{\partial\Sigma} h(x) < 0$;
3. $\mathfrak{D}_\tau(p) \neq 0$ *for any* $p \in \partial\Sigma$ *with* $\mathfrak{D}(p) = 1$.

*Then problem* (1.3) *admits a solution.*