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OPERS AND THE TWISTED BOGOMOLNY EQUATIONS

BY SIQI HE AND RAFE MAZZEO

ABSTRACT. – In this paper, we study the dimensionally reduced twisted Kapustin-Witten equations on the product of a compact Riemann surface Σ with $\mathbb{R}^+_y$. The main result is a Kobayashi-Hitchin type correspondence between the space of tilted Nahm pole solutions and the moduli space of Beilinson-Drinfeld opers. This corroborates a prediction of Gaiotto and Witten [*Adv. Theor. Math. Phys.* **16** (2012), p. 971].

RÉSUMÉ. – Dans cet article, nous étudions les équations de Kapustin-Witten tordues à dimensions réduites sur le produit d'une surface Riemann compacte Σ avec $\mathbb{R}^+$. Le résultat principal est une correspondance de type Kobayashi-Hitchin entre l'espace des solutions de pôles de Nahm inclinés et l'espace des modules des opérateurs de Beilinson-Drinfeld. Ceci corrobore une prédiction de Gaiotto et Witten [*Adv. Theor. Math. Phys.* **16** (2012), p. 971].

1. Introduction

Let (M, g) denote an oriented Riemannian 4-manifold and P a principal $SU(n)$ bundle over M with adjoint bundle $\mathfrak{g}_P$. The twisted Kapustin-Witten (TKW) equations [18] are a one-parameter family of equations, parametrized by $t \in (0, \infty)$, for a pair (A, Φ), where A is a connection and Φ is a $\mathfrak{g}_P$-valued 1-form:

$$F_A - \Phi \wedge \Phi + \frac{t - t^{-1}}{2} d_A \Phi + \frac{t + t^{-1}}{2} \star d_A \Phi = 0, \qquad d_A \star \Phi = 0. \tag{1}$$

When $t = 1$, these equations have the particularly simple form

$$F_A - \Phi \wedge \Phi + \star d_A \Phi = 0, \quad d_A \star \Phi = 0. \tag{2}$$

One important case is when $M = X \times \mathbb{R}^+_y$, where X is a 3-manifold. A fascinating proposal of Witten [29] interprets the Jones polynomial of knots in X by counting solutions to (2) which satisfy certain 'Nahm pole' singularities at $y = 0$, see [9, 30, 31] for a more detailed

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explanation, along with [22, 23, 11, 26, 19] for analytic theory related to this program. A similar program using the TKW Equations (1) to approach the Jones polynomial is also discussed in [29, 9]; a clearer formulation appears in [24], where the singular boundary condition appropriate for these equations when $t \neq 1$ is called the tilted Nahm pole boundary condition. We describe these below.

We consider here the dimensionally reduced TKW equations on $\Sigma \times \mathbb{R}^+_y$, where Σ is a compact Riemann surface. These are the TKW equations on $S^1 \times \Sigma \times \mathbb{R}^+_y$ for fields which are invariant in the S^1 direction. These fields consist of a connection A, a $\mathfrak{g}_P$-valued 1-form ϕ and $\mathfrak{g}_P$-valued 0-forms A_1 and ϕ_1; the corresponding 'twisted Bogomolny equations' take the form

$$(3)\qquad \begin{aligned} F_A - \phi\wedge\phi + \frac{t-t^{-1}}{2}d_A\phi - \frac{t+t^{-1}}{2}(\star d_A\phi_1 + \star[\phi, A_1]) = 0,\\ d_A A_1 - [\phi,\phi_1] + \frac{t-t^{-1}}{2}(d_A\phi_1 + [\phi, A_1]) - \frac{t+t^{-1}}{2}\star d_A\phi = 0,\\ d_A^\star\phi - [\phi_1, A_1] = 0. \end{aligned}$$

Write $t = \tan(\frac{\pi}{4}-\frac{3}{2}\beta)$. It was observed by Gaiotto and Witten in [9] that with the assumption $A_1 - \tan\beta\phi_1 = 0$, the twisted Bogomolny equations have a Hermitian-Yang-Mills structure, leading them to conjecture that there should be a Donaldson-Uhlenbeck-Yau type theorem in this setting. Such a result is now fully understood in the special case $\beta = 0$ ($t = 1$), cf. [12, 13, 14]. The results in those two papers give a precise correspondence in this spirit between flat $\mathrm{SL}(2,\mathbb{C})$ connections over Σ and solutions of the extended Bogomolny equations converging to these connections as $y\to\infty$ and satisfying certain singular boundary conditions at $y = 0$. More precisely, a flat $\mathrm{SL}(2,\mathbb{R})$ connection in the Hitchin section at infinity corresponds to solutions of the extended Bogomolny equations satisfying the Nahm pole boundary conditions at $y = 0$; an arbitrary stable Higgs pair (equivalently, a flat $\mathrm{SL}(2,\mathbb{C})$ connection), together with a holomorphic line subbundle, corresponds to a solution of the extended Bogomolny equations satisfying the Nahm pole boundary conditions with extra singularities along a divisor determined by the line bundle and the Higgs field. It is the generalization of the former of these two theorems which is the subject of this paper; a full generalization awaits a better understanding of the knot singularities for the tilted Nahm pole boundary conditions.

The study of (3) when $\beta \neq 0$ ($t \neq 1$) is motivated by the Atiyah-Floer approach to the Kapustin-Witten equations, see [1, 6] for recent progress. As pointed out in [9, Section 3], this Atiyah-Floer approach has physical obstructions and is unstable for the Equations (2) when $t = 1$, indicating that it may not be possible to recover the Jones polynomial entirely from that specialization of the equations. The paper [10] contains a more detailed explanation of this.

In any case, we consider here only the cases where $\beta \neq 0$. Denote by $\mathcal{M}^\beta_{\mathrm{TBE}}$ the space of solutions to the twisted Bogomolny equations with gauge group $SU(n)$ and with tilted Nahm pole conditions at $y = 0$ and a certain boundary condition to be explained later as $y\to\infty$. We denote by $\mathcal{M}^\beta_{\mathrm{Oper}}$ the twisted oper moduli space with parameter $\tan\beta$; this is diffeomorphic to the usual oper moduli space of Beilinson-Drinfield [3], and is defined in

Section 3. Using the Hermitian-Yang-Mills structure, Gaiotto and Witten [9] define

$$I^{\beta}_{\mathrm{Oper}} : \mathcal{M}^{\beta}_{\mathrm{TBE}} \to \mathcal{M}^{\beta}_{\mathrm{Oper}},$$

explained in Section 4 below, and predict that it is a bijection. We confirm their prediction here.

THEOREM 1.1. – *The map $I^{\beta}_{\mathrm{Oper}}$ is a bijection when the genus g of Σ is greater than* 1.

i) *For each element in $\mathcal{M}^{\beta}_{\mathrm{Oper}}$, there exists a solution to* (3) *with tilted Nahm pole singularity at $y = 0$;*

ii) *if two solutions satisfy tilted Nahm pole boundary condition and have the same image by $I^{\beta}_{\mathrm{Oper}}$, then they are gauge equivalent.*

There is an identification of $\mathcal{M}^{\beta}_{\mathrm{Oper}}$ with $\bigoplus_{i=2}^{n} H^0(K^i)$, where K is the canonical bundle over Σ, which gives a topology and differential structure to this space.

THEOREM 1.2. – *The map $I^{\beta}_{\mathrm{Oper}}$ is a diffeomorphism.*

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2. The Twisted Extended Bogomolny Equations

2.1. Hermitian Geometry for the Twisted Bogomolny Equations

We write the product metric on $\Sigma \times \mathbb{R}^+$, where Σ is a compact Riemann surface with genus $g \geq 1$, as $g = g_0^2|dz|^2 + dy^2$. Let E be a rank n complex vector bundle on this space with $\det E = 0$. An $SU(n)$ structure on E is determined by a Hermitian metric H. The adjoint bundle is denoted $\mathfrak{g}_E$.

Let A be a connection on E and suppose that $\phi \in \Omega^1(\mathfrak{g}_E)$ and $\phi_1 \in \Omega^0(\mathfrak{g}_E)$. In a unitary gauge defined by H, these satisfy $A^\star = -A, \phi^\star = -\phi, \phi_1^\star = -\phi_1$ where $\star$ is the conjugate transpose defined by H. Gaiotto and Witten observe in [9] that the twisted extended Bogomolny equations have a Hermitian Yang-Mills structure, and hence there should be a Donaldson-Uhlenbeck-Yau type result as in [7, 27]. Write $z = z_2 + iz_3$ for a local holomorphic coordinate on Σ and y for the linear coordinate on $\mathbb{R}^+$, then

$$d_A = \nabla_2 dx_2 + \nabla_3 dx_3 + \mathcal{D}_y dy, \text{ and } \phi = \phi_2 dx_2 + \phi_3 dx_3 = \frac{1}{2}(\phi_z dz + \phi_{\bar{z}} d\bar{z}).$$

Following [9, 29], define

$$\mathcal{D}_1 = (D_{\bar{z}} - \phi_{\bar{z}} \tan\beta) d\bar{z}, \qquad \mathcal{D}_2 = (D_z + \phi_z \cot\beta) dz, \qquad \mathcal{D}_3 = D_y - i\frac{\phi_1}{\cos\beta} \tag{4}$$

and their adjoints

$$\mathcal{D}_1^\dagger = (D_z - \phi_z \tan\beta) dz, \qquad \mathcal{D}_2^\dagger = (D_{\bar{z}} + \phi_{\bar{z}} \cot\beta) d\bar{z}, \qquad \mathcal{D}_3^\dagger = D_y + i\frac{\phi_1}{\cos\beta}. \tag{5}$$