ISSN 0012-9593

quatrième série - tome 55 fascicule 5 septembre-octobre 2022

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

Ariel RAPAPORT

Proof of the exact overlaps conjecture for systems with algebraic contractions

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Annales Scientifiques de l'École Normale Supérieure

Publiées avec le concours du Centre National de la Recherche Scientifique

Rédaction / *Editor*

Annales Scientifiques de l'École Normale Supérieure,
45, rue d'Ulm, 75230 Paris Cedex 05, France.
Tél. : (33) 1 44 32 20 88. Fax : (33) 1 44 32 20 80.
Email : `annales@ens.fr`

Édition et abonnements / *Publication and subscriptions*

Société Mathématique de France
Case 916 - Luminy
13288 Marseille Cedex 09
Tél. : (33) 04 91 26 74 64. Fax : (33) 04 91 41 17 51
Email : `abonnements@smf.emath.fr`

Tarifs

Abonnement électronique : 441 euros.
Abonnement avec supplément papier :
Europe : 619 €. Hors Europe : 698 € ($ 985). Vente au numéro : 77 €.

ISSN 0012-9593 (print) 1873-2151 (electronic)

Directeur de la publication : Fabien Durand
Périodicité : 6 nos / an

Ann. Scient. Éc. Norm. Sup.
4e série, t. 55, 2022, p. 1357 à 1377

PROOF OF THE EXACT OVERLAPS CONJECTURE FOR SYSTEMS WITH ALGEBRAIC CONTRACTIONS

BY Ariel RAPAPORT

ABSTRACT. – We establish the exact overlaps conjecture for iterated functions systems on the real line with algebraic contractions and arbitrary translations.

RÉSUMÉ. – Nous prouvons la conjecture de chevauchements exacts pour des systèmes de fonctions itérées définies sur l'ensemble des nombres réels à contractions algébriques et translations arbitraires.

1. Introduction

1.1. Background

Let $m \geq 1$ and $\Phi = \{\varphi_j(x) = \lambda_j x + t_j\}_{j=0}^m$ be a finite set of contracting similarities of $\mathbb{R}$, so that $0 \neq \lambda_j \in (-1, 1)$ and $t_j \in \mathbb{R}$ for each $0 \leq j \leq m$. Such a collection Φ is called a self-similar iterated function system (IFS). It is well known that there exists a unique nonempty compact $K \subset \mathbb{R}$, called the attractor of Φ, which satisfies the relation

$$K = \bigcup_{j=0}^{m} \varphi_j(K). \tag{1.1}$$

The set K is said to be self-similar.

Suppose additionally that $p = (p_j)_{j=0}^m$ is a probability vector. Then there exists a unique Borel probability measure $\mu = \mu(\Phi, p)$ on $\mathbb{R}$ such that

$$\mu = \sum_{j=0}^{m} p_j \cdot \varphi_j \mu,$$

where $\varphi_j \mu$ is the push-forward of μ by φ_j. The measure μ is supported on K, it is the unique stationary probability measure for the random walk moving from $x \in \mathbb{R}$ to $\varphi_j(x)$ with probability p_j, and it is called the self-similar measure corresponding to Φ and p. We shall always assume that p has strictly positive coordinates, in which case the support of μ is equal to K.

doi:10.24033/asens.2518

The dimension theory of self-similar measures is a central area of research in fractal geometry. It was proven by Feng and Hu [8] that μ is always exact dimensional. That is, there exists a value $\dim \mu \in [0,1]$, called the dimension of μ, such that

$$\dim \mu = \lim_{\delta \downarrow 0} \frac{\log \mu(x-\delta, x+\delta)}{\log \delta} \text{ for } \mu\text{-a.e. } x \in \mathbb{R}.$$

As proven in [6], $\dim \mu$ agrees with the value given to μ by other commonly used notions of dimension, such as the Hausdorff, packing and entropy dimensions.

It turns out that in most cases $\dim \mu$ satisfies a certain formula in terms of p and the contractions vector $\lambda = (\lambda_j)_{j=0}^m$. Denote by $H(p)$ the entropy of p and by χ the Lyapunov exponent corresponding to p and λ. That is,

$$H(p) = -\sum_{j=0}^{m} p_j \log p_j \text{ and } \chi = -\sum_{j=0}^{m} p_j \log |\lambda_j|, \tag{1.2}$$

where here and everywhere else in this paper the base of the log function is 2. Set,

$$\beta = \beta(\Phi, p) = \min\{1, H(p)/\chi\}, \tag{1.3}$$

then it is not hard to show that β is always an upper bound for $\dim \mu$ and that it is equal to $\dim \mu$ whenever the union in (1.1) is disjoint. Moreover, it was proven by Jordan, Pollicott and Simon [11] that if λ is kept fixed and $|\lambda_j| \in (0, \frac{1}{2})$ for each $0 \le j \le m$, then $\dim \mu = \beta$ for Lebesgue almost every selection of the translations $(t_j)_{j=0}^m \in \mathbb{R}^{m+1}$. A version of this result for sets was first established by Falconer [4].

There are cases in which it is obvious that dimension drop occurs, i.e., that $\dim \mu$ is strictly less than β. Denote the index set $\{0, \dots, m\}$ by Λ. For $n \ge 1$ and a word $j_1 \cdots j_n = w \in \Lambda^n$ set,

$$\varphi_w = \varphi_{j_1} \circ \cdots \circ \varphi_{j_n} \text{ and } \lambda_w = \lambda_{j_1} \cdots \lambda_{j_n}. \tag{1.4}$$

The IFS Φ is said to have exact overlaps if the semigroup generated by its elements is not free. Since the members of Φ are contractions, this is equivalent to the existence of $n \ge 1$ and distinct words $w_1, w_2 \in \Lambda^n$ with $\varphi_{w_1} = \varphi_{w_2}$. It is not difficult to see that $\dim \mu < \beta$ whenever Φ has exact overlaps and $\dim \mu < 1$. The following folklore conjecture says that these are the only circumstances in which dimension drop can occur. A version of it for sets was stated, probably for the first time, by Simon [16].

Conjecture 1. – *Suppose that* $\dim \mu < \beta$ *then* Φ *has exact overlaps.*

A major step towards the verification of Conjecture 1 was achieved by Hochman [9]. For $n \ge 1$ set,

$$\Delta_n = \min\left\{\left|\varphi_{w_1}(0) - \varphi_{w_2}(0)\right| \,:\, w_1, w_2 \in \Lambda^n,\ w_1 \ne w_2 \text{ and } \lambda_{w_1} = \lambda_{w_2}\right\}. \tag{1.5}$$

It always holds that $\Delta_n \overset{n}{\to} 0$ at a rate which is at least exponential, and that $\Delta_n = 0$ for some $n \ge 1$ if and only if Φ has exact overlaps. The main result in [9] says that if $\dim \mu < \beta$ then $\Delta_n \overset{n}{\to} 0$ super-exponentially, that is

$$\lim_n \frac{1}{n} \log \Delta_n = -\infty.$$

A version of this for L^q dimensions was recently obtained by Shmerkin [15, Theorem 6.6].

Two applications of Hochman's result are especially relevant to the present paper. It is not hard to see that if $\lambda_0, \ldots, \lambda_m, t_0, \ldots, t_m$ are all algebraic numbers and $\Delta_n \overset{n}{\to} 0$ super-exponentially, then in fact Φ must have exact overlaps. Relaying on this observation, Conjecture 1 is established in [9, Theorem 1.5] for the case of algebraic parameters. The second application verifies a conjecture of Furstenberg regarding projections of the one-dimensional Sierpinski gasket (see e.g., [13, Question 2.5]). Stated with the notation introduced above, it is proven in [9, Theorem 1.6] that Conjecture 1 is valid when $m = 2$ and

$$\lambda = p = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right).$$

Another important step towards Conjecture 1 was recently achieved by Varjú [17]. He has shown that if μ is a Bernoulli convolution, that is if in the notation above

$$m = 1,\ \lambda_0 = \lambda_1 > 0,\ t_0 = -1 \text{ and } t_1 = 1,$$

then $\dim \mu = \beta$ whenever λ_0 is transcendental. Together with the result mentioned above regarding systems with algebraic parameters, this verifies Conjecture 1 for the family of Bernoulli convolutions.

Given Hochman's and Shmerkin's results, it is natural to ask whether Φ has exact overlaps whenever $\Delta_n \overset{n}{\to} 0$ super-exponentially. Recently, examples have been constructed by Baker [1] and independently by Bárány and Käenmäki [2], which show that this is not necessarily true. In Baker's construction the maps in the IFS all contract by a rational number, and so it is especially relevant to the present paper. In a joint work with P. Varjú [14] we will treat a family of self-similar measures which is closer to the example from [2].

1.2. Results

The following theorem is our main result. It verifies Conjecture 1 for the case of algebraic contractions and arbitrary translations.

THEOREM 2. – *Let $m \geq 0$ and $\Phi = \{\varphi_j(x) = \lambda_j x + t_j\}_{j=0}^m$ be a self-similar IFS on $\mathbb{R}$. Suppose that $\lambda_0, \ldots, \lambda_m$ are all algebraic numbers and that Φ has no exact overlaps. Let $p = (p_j)_{j=0}^m$ be a probability vector and denote by μ the self-similar measure corresponding to Φ and p. Then* $\dim \mu = \beta$, *where β is as defined in (1.3).*

A version for sets of the conjecture follows directly from the last theorem in the case of algebraic contractions. Given an IFS Φ as above denote by $\dim_s \Phi$ its similarity dimension, that is $\dim_s \Phi$ is the unique $s \geq 0$ which satisfies the equation

$$\sum_{j=0}^{m} |\lambda_j|^s = 1.$$

It is not hard to see that $\min\{1, \dim_s \Phi\}$ is always an upper bound for $\dim_H K$, where K is the attractor of Φ and $\dim_H$ stands for Hausdorff dimension. Moreover, the equality

$$\dim_H K = \min\{1, \dim_s \Phi\} \tag{1.6}$$

is satisfied when the union in (1.1) is disjoint or, more generally, if Φ satisfies the so-called open set condition (see for instance [3, Chapter 2.1]). The version for sets of Conjecture 1 says that (1.6) holds whenever Φ has no exact overlaps.