ISSN 0012-9593

*quatrième série - tome 55* *fascicule 5* *septembre-octobre 2022*

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

Mladen BESTVINA & Vincent GUIRARDEL & Camille HORBEZ

# *Boundary amenability of* $\mathbf{Out}(F_N)$

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

# Annales Scientifiques de l'École Normale Supérieure

Publiées avec le concours du Centre National de la Recherche Scientifique



**Rédaction / *Editor***

Annales Scientifiques de l'École Normale Supérieure,
45, rue d'Ulm, 75230 Paris Cedex 05, France.
Tél. : (33) 1 44 32 20 88. Fax : (33) 1 44 32 20 80.
Email : annales@ens.fr

---

**Édition et abonnements / *Publication and subscriptions***

Société Mathématique de France
Case 916 - Luminy
13288 Marseille Cedex 09
Tél. : (33) 04 91 26 74 64. Fax : (33) 04 91 41 17 51
Email : abonnements@smf.emath.fr

**Tarifs**

Abonnement électronique : 441 euros.
Abonnement avec supplément papier :
Europe : 619 €. Hors Europe : 698 € ($ 985). Vente au numéro : 77 €.

---



---

ISSN 0012-9593 (print) 1873-2151 (electronic)

Directeur de la publication : Fabien Durand
Périodicité : 6 nos / an

*Ann. Scient. Éc. Norm. Sup.*
4e série, t. 55, 2022, p. 1379 à 1431

# BOUNDARY AMENABILITY OF Out($F_N$)

BY Mladen BESTVINA, Vincent GUIRARDEL YYY
AND Camille HORBEZ

*Kaum nennt man die Dinge beim richtigen Namen,*
*so verlieren sie ihren gefährlichen Zauber.*
Elias Canetti

*What's in a name? That which we call a rose*
*By any other name would smell as sweet.*
William Shakespeare

*Mal nommer un objet,*
*c'est ajouter au malheur de ce monde.*
Albert Camus

Abstract. – We prove that Out($F_N$) is boundary amenable. This also holds more generally for Out($G$), where $G$ is either a toral relatively hyperbolic group or a finitely generated right-angled Artin group. As a consequence, all these groups satisfy the Novikov conjecture on higher signatures.

Résumé. – Nous montrons que Out($F_N$) est moyennable à l'infini. Plus généralement, si $G$ est un groupe relativement hyperbolique torique ou un groupe d'Artin à angles droits de type fini, alors Out($G$) est moyennable à l'infini. En conséquence, dans chacun de ces cas, le groupe Out($G$) satisfait la conjecture de Novikov.

## 1. Introduction

Boundary amenability—also known as *exactness* or *coarse amenability*, and also equivalent to Yu's *Property A* from [55] (as was shown by Higson and Roe in [27])—is a property of a countable group that has important applications in K-theory, operator algebras and measured group theory. The reader is referred to [2, 47] for general introductions. The definition is as follows.

 doi:10.24033/asens.2519

Definition 1.1 (Boundary amenability). – *A countable discrete group* $\Gamma$ *is* boundary amenable *if there exist a nonempty compact Hausdorff space* $X$ *equipped with an action of* $\Gamma$ *by homeomorphisms and a sequence of continuous maps*

$$\mu_n : X \to \mathrm{Prob}(\Gamma)$$

*such that for all* $\gamma \in \Gamma$, *one has*

$$\sup_{x \in X} ||\mu_n(\gamma.x) - \gamma.\mu_n(x)||_1 \to 0,$$

*as* $n$ *goes to* $+\infty$.

In this definition, $\mathrm{Prob}(\Gamma)$ denotes the space of probability measures on $\Gamma$, equipped with the topology of pointwise convergence, or equivalently, subspace topology from $\ell^1(\Gamma)$—the continuity of the maps $\mu_n$ in the above definition is understood with respect to this topology. An action $\Gamma \curvearrowright X$ as in Definition 1.1 is called *topologically amenable*.

Boundary amenability has already been established for several important classes of groups. Guentner, Higson and Weinberger proved in [17] that all linear groups are boundary amenable. Campbell and Niblo proved in [9] that every group acting properly and cocompactly on a finite-dimensional CAT(0) cube complex is boundary amenable. Boundary amenability is also known for many groups satisfying 'hyperbolic-like' properties: this was established by Adams [1] for hyperbolic groups and extended by Ozawa [48] to the case of relatively hyperbolic groups with boundary amenable parabolic subgroups. It was then established for mapping class groups of orientable surfaces of finite type by Kida [34] and Hamenstädt [25], and for automorphism groups of locally finite buildings by Lécureux [37]. On another note, finitely generated groups whose Cayley graphs contain a properly embedded expander are not boundary amenable; examples of such groups were constructed by Gromov [16] (see also [4]), and more recently Osajda constructed residually finite examples [45]. We also mention that work of Arzhantseva, Guentner and Spakula [5] provides examples of non-coarsely amenable metric spaces of different nature.

The goal of the present paper is to establish the boundary amenability of $\mathrm{Out}(F_N)$, the outer automorphism group of a finitely generated free group $F_N$. This group has been the subject of intensive research for a while, see e.g., [54] for a recent survey. More generally, we prove the following theorem.

Theorem 1.2 (see Corollaries 5.5 and 5.7). – *Let a finitely generated group G be either*

1. *a free group,*
2. *a torsion-free Gromov hyperbolic group,*
3. *a torsion-free toral relatively hyperbolic group,*
4. *a right-angled Artin group.*

*Then* $\mathrm{Out}(G)$ *is boundary amenable.*

Since $\mathrm{Aut}(G)$ embeds in $\mathrm{Out}(G * \mathbb{Z})$ and boundary amenability is stable under taking subgroups, boundary amenability of $\mathrm{Aut}(G)$ follows in these cases:

Corollary 1.3. – *Let* $G$ *be a group as in Theorem 1.2. Then* $\mathrm{Aut}(G)$ *is boundary amenable.*

We expect that the torsion-freeness assumption in Theorem 1.2 is not necessary. But if $G$ is a virtually torsion-free hyperbolic group, one can deduce from Theorem 1.2 that $\mathrm{Out}(G)$ and $\mathrm{Aut}(G)$ are boundary amenable, see Corollary 5.6 (whether or not there exists a hyperbolic group which is not virtually torsion-free is a famous open question).

When $G = F_N$, an explicit compact space equipped with a topologically amenable action of $\mathrm{Out}(F_N)$ is constructed in Section 6.1 of the present paper: this space is an infinite-dimensional product space involving all boundaries of relative Outer spaces associated to free factors $A \subseteq F_N$ and free factor systems of $A$, and all boundaries of free factors of $F_N$. But our general proof strategy does not consist in working directly with this compact space; instead we rely on an inductive argument coming from Ozawa's work on boundary amenability of relatively hyperbolic groups [48], as will be explained later in this introduction.

Notice that every identification between the fundamental group of a surface $\Sigma$ obtained from a closed connected surface by removing a finite non-empty set of points, and a finitely generated free group $F_N$, yields an embedding of the mapping class group of $\Sigma$ into $\mathrm{Out}(F_N)$. Since boundary amenability passes to subgroups, this gives a new proof of the boundary amenability of the mapping class group of every punctured surface $\Sigma$ as above.

*Applications*. – A key motivation behind the study of boundary amenability comes from a theorem that follows from work of Yu [55], Higson-Roe [27] and Higson [26], stating that boundary amenability of $\Gamma$ implies the injectivity of the Baum-Connes assembly map, which in turn implies the Novikov conjecture on higher signatures for $\Gamma$ (this theorem builds on the fact that boundary amenability of $\Gamma$ is equivalent to $\Gamma$ satisfying Yu's property A, which implies in turn that $\Gamma$ admits a coarse embedding in a Hilbert space). Since boundary amenability passes to subgroups [47], we get the following corollary to Theorem 1.2.

COROLLARY 1.4. – *Let $G$ be a group as in Theorem 1.2. Then* $\mathrm{Out}(G)$ *and any of its subgroups satisfy the Novikov conjecture.*

Another application of the boundary amenability of a group $\Gamma$ comes from the study of certain operator algebras associated to $\Gamma$: for example, boundary amenability of a countable group is equivalent to the exactness of its reduced $C^*$-algebra, see [2, 46].

*Boundary amenability of the automorphism group of a free product*. – In order to establish Theorem 1.2, we actually work in the more general setting of groups coming with a decomposition as a free product. Let $k, N$ be non-negative integers, let $\{G_1, \dots, G_k\}$ be a finite family of countable groups, and let

$$G := G_1 * \cdots * G_k * F_N.$$

We let $\mathcal{F} = \{G_1, \dots, G_k\}$, naturally viewed as a family of subgroups of $G$. We denote by $\mathrm{Out}(G, \mathcal{F})$ the subgroup of $\mathrm{Out}(G)$ made of all automorphisms which preserve the conjugacy class of each subgroup $G_i$, and by $\mathrm{Out}(G, \mathcal{F}^{(t)})$ the subgroup made of all automorphisms that act as the conjugation by an element $g_i \in G$ on each subgroup $G_i$. Our main theorem is the following:

THEOREM 1.5 (see Theorem 5.2). – *Let $k, N$ be non-negative integers.*
*Let $\mathcal{F} = \{G_1, \dots, G_k\}$ be a finite family of countable groups, and let*

$$G := G_1 * \cdots * G_k * F_N.$$