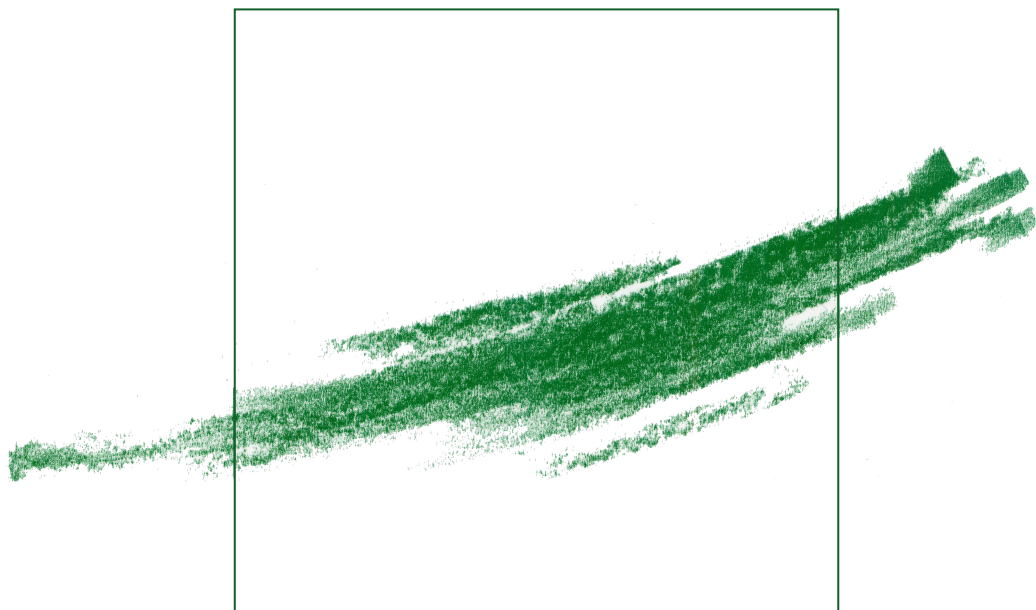


COURS SPÉCIALISÉS  
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# Integral geometry from Buffon to geometers of today

Rémi LANGEVIN



23



**INTEGRAL GEOMETRY FROM BUFFON  
TO GEOMETERS OF TODAY**

**Rémi Langevin**

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**Société Mathématique de France 2015**



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