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Compact Quantum Groups and Their Representation Categories

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PREFACE

The term 'quantum group' was popularized in the 1980s and, in fact, does not have a precise meaning. The closely related, and rigorously defined, notion of a Hopf algebra appeared much earlier, in the 1950s. It has its origin in a work by Hopf [**41**] from 1941 on algebraic topology, who observed that the cohomology ring of a compact group G has a homomorphism $H^*(G) \to H^*(G) \otimes H^*(G)$. A related, and even more elementary, example of such structure is the following: for a finite group G and the algebra $C(G)$ of functions on G with pointwise multiplication we can define a homomorphism, called comultiplication,

$$\Delta\colon C(G) \to C(G) \otimes C(G) = C(G \times G) \ \text{ by } \ \Delta(f)(g,h) = f(gh).$$

What is important, is that the pair $(C(G), \Delta)$ contains complete information about the group G: the spectrum of the algebra $C(G)$ is G, and the comultiplication Δ allows us to recover the group law. We refer the reader to [**1**] for a thorough discussion of the origins of the theory of Hopf algebras. The part of the story that is particularly relevant for us starts in the early 1960s with a work by Kac [**47**]. His idea was to develop a duality theory that generalizes Pontryagin duality for abelian locally compact groups. Such a generalization for compact groups had already been obtained by Tannaka [**77**] and Krein [**54**], but even in that case it was not entirely satisfactory in the sense that the dual of a compact group G was an object of a quite different nature, the category of finite dimensional representations of G concretely realized as a category of vector spaces. Kac's idea was to describe both a locally compact group and its dual using von Neumann algebras with comultiplication satisfying certain properties, and this way obtain a self-dual category. Such a theory, nowadays called the theory of Kac algebras, was finally developed in the 1970s by Kac-Vainerman and Enock-Schwartz, see [**29**].

Being a significant technical achievement, the theory of Kac algebras nevertheless suffered from the lack of interesting examples that were not of group origin, that is, were neither algebras of functions nor their duals, group algebras. For similar reasons the general theory of Hopf algebras remained at that time a small branch of algebra. The situation changed drastically in the middle 1980s, when Jimbo [**44**] and Drinfeld [**26**] introduced new Hopf algebras by deforming universal enveloping algebras of semisimple Lie groups. Working in the formal deformation setting Drinfeld also introduced their dual objects, deformations of the Hopf algebras of regular functions on semisimple Lie groups. He suggested the term 'quantum groups' for Hopf algebras related to these constructions. In the analytic, non-formal, setting the quantized algebra of functions on $SU(2)$ was then studied in detail by Vaksman and Soibelman [**82**]. Simultaneously, and independently of Drinfeld-Vaksman-Soibelman, a deformation of the algebra of continuous functions on $SU(2)$ was defined by Woronowicz [**95**]. Remarkably, he arrived at exactly the same definition.

Following the foundational works by Drinfeld, Jimbo, Soibelman, Vaksman and Woronowicz, the theory of quantum groups saw several years of explosive growth, apparently unprecedented in the history of mathematics, with ground-breaking applications to knot theory, topology of 3-manifolds and conformal field theory [**80**]. This was entwined with the development of noncommutative geometry by Connes, the free probability theory by Voiculescu and the Jones theory of subfactors. Since then quantum group theory has developed in several directions and by now there is probably no single expert who has a firm grasp of all of its aspects.

In this book we take the analytic point of view, meaning that we work with algebras of, preferably bounded, operators on Hilbert spaces. For an introduction to quantum groups from the purely algebraic side see e.g., [**18**]. According to the standard mantra of noncommutative geometry, an algebra of operators should be thought of as an algebra of functions on a noncommutative locally compact space, with C*-algebras playing the role of continuous functions and von Neumann algebras playing the role of measurable functions. From this perspective a quantum group is a C*-/von Neumann algebra with some additional structure making the noncommutative space a group-like object. Kac algebras give an example of such structure, but as it turned out their class is too narrow to accommodate the objects arising from Drinfeld-Jimbo deformations. A sufficiently broad theory was developed first in the compact case by Woronowicz [**97**], and then in the general, significantly more complicated, locally compact case by Kustermans-Vaes [**55**] and Masuda-Nakagami-Woronowicz [**63**]. No theory is complete without interesting examples, and here there are plenty of them. In addition to examples arising from Drinfeld-Jimbo deformations, there is a large class of quantum groups defined as symmetries of noncommutative spaces. This line of research was initiated by Wang [**89, 90**] and has been extensively pursued by Banica and his collaborators, see e.g., [**8, 9**]. A related idea is to define quantum isometries of noncommutative Riemannian manifolds, recently suggested by Goswami and Bhowmick [**37, 12**].

The goal of this short book is to introduce the reader to this beautiful area of mathematics, concentrating on the technically easier compact case and emphasizing the role of the categorical point of view in constructing and analyzing concrete examples. Specifically, the first two chapters, occupying approximately 2/3 of the book, contain a general theory of compact quantum groups together with some of the most famous examples. Having mastered the material in these chapters, the reader will hopefully be well prepared for a more thorough study of any of the topics we mentioned above. The next two chapters are motivated by our own interests in noncommutative geometry of quantum groups and concentrate on certain aspects of the structure of Drinfeld-Jimbo deformations. The general theme of these chapters is the Drinfeld-Kohno theorem, which is one of the most famous results in the whole theory of quantum groups, presented from the analytic point of view together with its operator algebraic ramifications. Each section is supplied with a list of references. We try to give references to original papers, where the results of a particular section and/or some related results have appeared. The literature on quantum groups is vast, so some omissions are unavoidable, and the references are meant to be pointers to the literature rather than exhaustive bibliographies on a particular subject.

We tried to make the exposition reasonably self-contained, but certain prerequisites are of course assumed. The book is first of all intended for students specializing in operator algebras, so we assume that the reader has at least taken a basic course in C*-algebras as e.g., covered in Murphy [**65**]. The reader should also have a minimal knowledge of semisimple Lie groups, see e.g., Part II in Bump [**17**], without which it is difficult to fully understand Drinfeld-Jimbo deformations. Finally, it is beneficial to have some acquaintance with category theory. Although we give all the necessary definitions in the text (apart from the most basic ones, for those see e.g., the first chapter in Mac Lane [**61**]), the reader who sees them for the first time will have to work harder to follow the arguments.

Let us say a few words about notation.

We denote by the same symbol $\otimes$ all kinds of tensor products, the exact meaning should be clear from the context: for spaces with no topology this denotes the usual tensor product over $\mathbb{C}$, for Hilbert spaces - the tensor product of Hilbert spaces (that is, the completion of the algebraic tensor product with respect to the obvious scalar product), for C*-algebras - the *minimal* tensor product.

For vector spaces H_1 and H_2 with no topology we denote by $B(H_1, H_2)$ the space of linear operators $H_1 \to H_2$. If H_1 and H_2 are Hilbert spaces, then the same symbol $B(H_1, H_2)$ denotes the space of bounded linear operators. We write $B(H)$ instead of $B(H, H)$.

If $\mathcal{A}$ is a vector space with no topology, then $\mathcal{A}^*$ denotes the space of all linear functionals on $\mathcal{A}$. For topological vector spaces the same symbol denotes the space of all continuous linear functionals.

The symbol ι denotes the identity map.

In order to simplify long complicated expressions, we omit the symbol $\circ$ for the composition of maps, as well as use brackets only for arguments, but not for maps. Thus we write $ST(x)$ instead of $(S \circ T)(x)$.

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