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Eusebio GARDELLA & Hannes THIEL

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Annales Scientifiques de l'École Normale Supérieure,
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Tél. : (33) 1 44 32 20 88. Fax : (33) 1 44 32 20 80.
Email : annales@ens.fr

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Société Mathématique de France
Case 916 - Luminy
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Tél. : (33) 04 91 26 74 64. Fax : (33) 04 91 41 17 51
Email : abonnements@smf.emath.fr

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ISOMORPHISMS OF ALGEBRAS OF CONVOLUTION OPERATORS

BY Eusebio GARDELLA AND Hannes THIEL

ABSTRACT. – For $p, q \in [1, \infty)$, we study the isomorphism problem for the p-and q-convolution algebras associated to locally compact groups. While it is well known that not every group can be recovered from its group von Neumann algebra, we show that this is the case for the algebras $\mathrm{CV}_p(G)$ of p-convolvers and $\mathrm{PM}_p(G)$ of p-pseudomeasures, for $p \neq 2$. More generally, we show that if $\mathrm{CV}_p(G)$ is isometrically isomorphic to $\mathrm{CV}_q(H)$, with $p, q \neq 2$, then G must be isomorphic to H and p and q are either equal or conjugate. This implies that there is no L^p-version of Connes' uniqueness of the hyperfinite II_1-factor. Similar results apply to the algebra $\mathrm{PF}_p(G)$ of p-pseudofunctions, generalizing a classical result of Wendel. We also show that other L^p-rigidity results for groups can be easily recovered and extended using our main theorem.

Our results answer questions originally formulated in the work of Herz in the 70's. Moreover, our methods reveal new information about the Banach algebras in question. As a non-trivial application, we verify the reflexivity conjecture for all Banach algebras lying between $\mathrm{PF}_p(G)$ and $\mathrm{CV}_p(G)$: if any such algebra is reflexive and amenable, then G is finite.

RÉSUMÉ. – Pour $p, q \in [1, \infty)$, nous étudions le problème d'isomorphisme pour les algèbres de convolution L^p et L^q associées à un groupe localement compact. S'il est bien connu qu'on ne peut en général pas retrouver le groupe à partir de son algèbre de von Neumann, nous montrons que c'est le cas pour les algèbres $\mathrm{CV}_p(G)$ de p-convoluteurs et $\mathrm{PM}_p(G)$ de p-pseudomesures, pour $p \neq 2$. Plus généralement, nous montrons que si $\mathrm{CV}_p(G)$ est isométriquement isomorphe à $\mathrm{CV}_q(H)$ avec $p, q \neq 2$, alors G est nécessairement isomorphe à H et p et q sont égaux ou conjugués. Cela implique qu'il n'y a pas de version L^p de l'unicité, due à Connes, du facteur II_1 hyperfini. Ces résultats s'appliquent aussi aux algèbres $\mathrm{PF}_p(G)$ de p-pseudofunctions, et généralisent un résultat classique de Wendel. Nous montrons également que d'autres résultats de rigidité L^p pour les groupes peuvent être retrouvés et étendus à l'aide de notre résultat principal.

Nos travaux répondent à des questions initialement posées dans les travaux de Herz dans les années 1970. De plus, nos méthodes révèlent de nouvelles informations sur les algèbres de Banach en question. Comme application non triviale, nous vérifions la conjecture de réflexivité pour toutes les algèbres de

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Banach situées entre $\mathrm{PF}_p(G)$ et $\mathrm{CV}_p(G)$: si une telle algèbre est réflexive et moyennable, alors G est fini.

1. Introduction

Convolution algebras of groups are among the most important and widely studied examples of Banach algebras. For a locally compact group G, the algebras that have arguably received the greatest attention are the function algebra $L^1(G)$, the measure algebra $M^1(G)$, the reduced group C*-algebra $C^*_\lambda(G)$, and the group von Neumann algebra $L(G)$. A significant part of the literature in this area has focused on identifying those properties of a group that are reflected on its convolution algebras. An early and illustrative instance of this is Johnson's celebrated result [29] asserting that a locally compact group G is amenable if and only if $L^1(G)$ is amenable as a Banach algebra. Similar results have been obtained for $M^1(G)$, $C^*_\lambda(G)$ and $L(G)$, at least when G is discrete.

The isomorphism problem in Harmonic Analysis asks to determine when two groups have isometrically isomorphic convolution algebras. This problem has received a great deal of attention, particularly in what refers to identifying those groups that can be *recovered* from one of their convolution algebras. The first result in this direction is Wendel's classical result [43], asserting that $L^1(G)$ is isometrically isomorphic to $L^1(H)$ if and only if G is (topologically) isomorphic to H. Shortly after, Johnson used Wendel's theorem to prove a similar result for the measure algebra. The situation for the operator algebras $L(G)$ and $C^*_\lambda(G)$ is, however, more complicated. For once, not every group can be recovered from its von Neumann algebra, or even from its reduced group C*-algebra: consider, for example, $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ and $\mathbb{Z}_4$. More drastically, Connes' celebrated result on uniqueness of the hyperfinite II_1-factor implies that any two countable amenable ICC groups have isomorphic group von Neumann algebras. There also exist groups that can be recovered from their C^*-algebras but not from their von Neumann algebras (such as $\mathbb{Z}$). Positive results in this context (usually referred to as *superrigidity* results) are difficult to find: for von Neumann algebras, the first one is the groundbreaking work of Ioana-Popa-Vaes ([28]) on wreath product groups, while non-trivial results for C^*-algebras are even more recent ([9, 31]). Indeed, the passage from $C^*_\lambda(G)$ to $L(G)$ tends to "erase" a lot of information about G. For example, while every countable, torsion-free, abelian group is recovered from its reduced group C^*-algebra, all such groups have isomorphic group von Neumann algebras. Similarly, while it is known that $\mathbb{F}_n$ and $\mathbb{F}_m$ have non-isomorphic group C^*-algebras, whether this is the case for their von Neumann algebras is a notable open problem.

In this work, we are interested in the L^p-version of the problems described above. Given $p \in [1, \infty)$ and a locally compact group G, we let λ_p be the representation of $L^1(G)$ on $L^p(G)$ given by left convolution. The algebra of *p-pseudofunctions* $\mathrm{PF}_p(G)$ is the Banach algebra generated by $\lambda_p(L^1(G))$; the algebra of *p-convolvers* $\mathrm{CV}_p(G)$ is the double commutant of $\mathrm{PF}_p(G)$; and, for $p > 1$, the algebra of *p-pseudomeasures* $\mathrm{PM}_p(G)$ is the weak*-closure [(1)] of $\mathrm{PF}_p(G)$. We thus obtain a continuously varying family of Banach algebras which for $p = 1$ give the algebras $L^1(G)$ and $M^1(G)$, and for $p = 2$ agree with $C^*_\lambda(G)$ and $L(G)$.

These algebras were introduced by Herz [27] several decades ago, and have been intensively studied by a number of authors; see [8, 7]. In recent years, the influx of operator-algebraic techniques has given the area new impetus; see [37, 26, 20, 22, 19, 18, 2], and see also [17] for a recent survey on the topic. Among these, we mention the solution [21] to a long-standing open problem of Le Merdy: for $p \neq 2$, there exists a quotient of $\mathrm{PF}_p(\mathbb{Z})$ which cannot be represented on an L^p-space.

Despite the advances, a number of questions remain open, not least due to the difficulties in understanding the geometry of L^p-spaces (by comparison with the case $p = 2$). A significant problem in the area, known as the *convolvers and pseudomeasures problem* and originally raised by Herz, asks to determine if the double-commutant theorem holds for $\mathrm{PM}_p(G)$, that is, whether it is always true that $\mathrm{PM}_p(G) = \mathrm{CV}_p(G)$. This is known to be the case for $p = 2$ (regardless of G), and whenever G has the approximation property (regardless of p); see [4, 7].

Another problem, also raised by Herz, is the isomorphism question for these algebras. In its most general form, the problem is to determine, for $p, q \in [1, \infty)$ and locally compact groups G and H, when there exists an isometric isomorphism $\mathrm{CV}_p(G) \cong \mathrm{CV}_q(H)$ (or $\mathrm{PM}_p(G) \cong \mathrm{PM}_q(H)$, or $\mathrm{PF}_p(G) \cong \mathrm{PF}_q(H)$). Earlier results [27, 8] mostly focused on duality theory and offered partial answers when $G = H$; this case was recently settled by the authors in [22]. In this work, we focus on the remaining and arguably most difficult part of this problem, namely deciding when two groups have isometrically isomorphic p-convolution algebras.

The outcome is not a priori clear. Indeed, p-convolution algebras tend to behave more like the case $p = 2$ (group C^*-algebras and von Neumann algebras) than like the case $p = 1$ (function and measure algebras), mostly thanks to either reflexivity, uniform convexity, or interpolation for operators on L^p-spaces. Some examples of this are as follows:

— For a Powers group G and $p \in (1, \infty)$, the Banach algebra $\mathrm{PF}_p(G)$ is simple, while this fails for $p = 1$; see [26].

— For $p \in (1, \infty)$, amenability of G is characterized by the fact that $\mathrm{PF}_p(G)$ is universal with respect to representations of G on L^p-spaces, while this fails for $p = 1$; see [20].

— For a discrete group G with trivial amenable radical and $p \in (1, \infty)$, the Banach algebras $\mathrm{PF}_p(G)$, $\mathrm{PM}_p(G)$ and $\mathrm{CV}_p(G)$ have a unique (weak*-continuous) tracial state [(2)], while this fails for $p = 1$.

(1) Identifying $\mathcal{B}(L^p(G))$ with the dual of $L^p(G)\widehat{\otimes}L^{p'}(G)$ in a canonical way.

(2) Since this result is not available in the literature, we outline its proof here. By Theorem 5.2 in [25], triviality of the amenable radical is equivalent to a norm inequality in $\mathcal{B}(\ell^2(G))$ for a convex combination of unitaries coming from the left regular representation. Using the Riesz-Thorin interpolation theorem, it is easy to see that this condition is equivalent to an identical norm inequality in $\mathcal{B}(\ell^p(G))$, for $p \in (1, \infty)$. Once this is established, it follows that any linear functional on $\mathrm{PF}_p(G)$ which is "unitary invariant" must vanish on the non-trivial group elements. From this, uniqueness of the trace on $\mathrm{PF}_p(G)$ can be deduced immediately. The arguments for weak*-continuous traces on $\mathrm{CV}_p(G)$ and $\mathrm{PM}_p(G)$ are analogous.