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GEODESIC RAYS AND STABILITY IN THE cscK PROBLEM

BY CHI LI

ABSTRACT. — We prove that any finite energy geodesic ray with a finite Mabuchi slope is maximal in the sense of Berman-Boucksom-Jonsson, and reduce the proof of the uniform Yau-Tian-Donaldson conjecture for constant scalar curvature Kähler metrics to Boucksom-Jonsson's regularization conjecture about the convergence of non-Archimedean entropy functional. As further applications, we show that a uniform K-stability condition for model filtrations and the \mathcal{J}^{K_X} -stability are both sufficient conditions for the existence of cscK metrics. The first condition is also conjectured to be necessary. Our arguments also produce a different proof of the toric uniform version of YTD conjecture for all polarized toric manifolds. Another result proved here is that the Mabuchi slope of a geodesic ray associated to a test configuration is equal to the non-Archimedean Mabuchi invariant.

RÉSUMÉ. — Nous démontrons que tout rayon géodésique d'énergie finie de pente de Mabuchi finie est maximal au sens de Berman-Boucksom-Jonsson, et réduisons la preuve de la conjecture de Yau-Tian-Donaldson uniforme pour les métriques kähleriennes de courbure scalaire constante à la conjecture de régularisation de Boucksom-Jonsson sur la convergence de la fonctionnelle d'entropie non archimédienne. Nous montrons, comme autres applications, qu'une condition de K-stabilité uniforme pour les filtrations modèles et la \mathcal{J}^{K_X} -stabilité sont toutes les deux des conditions suffisantes pour l'existence de métriques cscK. La première condition est également conjecturée être nécessaire. Nos arguments fournissent aussi une preuve différente de la version uniforme torique de la conjecture YTD pour toutes les variétés toriques polarisées. Un autre résultat obtenu ici est que la pente de Mabuchi d'un rayon géodésique associée à un test de configuration est égale à l'invariant de Mabuchi non archimédien.

1. Introduction and main results

Let (X, L) be a polarized algebraic manifold and $[\omega] = c_1(L) > 0$ be the Hodge class. The Yau-Tian-Donaldson (YTD) conjecture aims to give a sufficient and necessary algebraic condition for the existence of constant scalar curvature Kähler (cscK) metric in the Kähler class $[\omega]$. Recently there have been significant progresses towards this conjecture, especially

on the analytic part (see [7, 8, 23, 24, 25, 32]) and the Fano case [26, 63, 35]. Also, Berman-Boucksom-Jonsson [2, 6, 10] proposed a variational approach for attacking this conjecture, which has been successfully carried out in the Fano case, even for singular Fano varieties (see [6, 47, 49, 51] and Section 6.1).

Geodesic rays play important roles in the recent study of cscK problem. Indeed, it has been shown that the non-existence of cscK metric is equivalent to the existence of non-trivial destabilizing geodesic rays (see [25, 6, 32]). We recall the following definition and refer to Section 2.2 for definition of geodesic rays and (99) for the expression of Mabuchi energy \mathbf{M} .

DEFINITION 1.1. – *For a finite energy geodesic ray $\Phi = \{\varphi(s)\} : \mathbb{R}_{\geq 0} \rightarrow \mathcal{E}^1(L)$, its Mabuchi slope is:*

$$(1) \quad \mathbf{M}'^\infty(\Phi) := \lim_{s \rightarrow +\infty} \frac{\mathbf{M}(\varphi(s))}{s}.$$

We say that Φ is destabilizing if $\mathbf{M}'^\infty(\Phi) \leq 0$.

The existence of the above limit, which may be $+\infty$, follows from the convexity of \mathbf{M} along Φ as proved in [7] based on the convexity along $C^{1,\bar{1}}$ -geodesics proved earlier in [3].

To make contact with the YTD conjecture, we would like to know whether destabilizing geodesic rays are algebraically approximable, meaning whether they can be approximated by a decreasing sequence of geodesic rays associated to test configurations. Such geodesic rays are maximal in the sense of Berman-Boucksom-Jonsson [6]. It was shown by Berman-Boucksom-Jonsson [6] that there is a one-to-one correspondence between finite energy non-Archimedean metrics and maximal geodesic rays (see Theorem 2.34).

Our first result says that any destabilizing geodesic rays are automatically algebraically approximable from above, i.e., maximal.

THEOREM 1.2. – *Let $\Phi : \mathbb{R}_{\geq 0} \rightarrow \mathcal{E}^1(L)$ be a geodesic ray. Assume that $\mathbf{M}'^\infty(\Phi) < \infty$. Then Φ is maximal. As a consequence, we have the identity:*

$$(2) \quad \mathbf{E}'^\infty(\Phi) = \mathbf{E}^{\text{NA}}(\Phi_{\text{NA}}).$$

To solve the YTD conjecture it remains to show that Mabuchi slopes of destabilizing (maximal) geodesic rays are algebraically approximable, which means that they can be approximated by Mabuchi slopes of geodesic rays associated to test configurations. By Chen-Tian's formula in (99), the Mabuchi energy has a decomposition into the entropy part and the energy part, and the energy part can be decomposed into the sum of Monge-Ampère and twisted Monge-Ampère energy. The slopes of Monge-Ampère energy for maximal geodesics can be algebraically approximated (see Theorem 2.34). The following result shows that the twisted Monge-Ampère slopes of maximal geodesic rays are also algebraically approximable. In this paper $\mathcal{H}^{\text{NA}}(L)$ always denotes the set of (smooth positive) non-Archimedean metrics that are associated to semiample test configurations of (X, L) .

THEOREM 1.3. – *Let (Q, ψ_Q) be a line bundle over X with a smooth Hermitian metric $e^{-\psi_Q}$. Assume that $\Phi : \mathbb{R}_{\geq 0} \rightarrow \mathcal{E}^1(L)$ is a maximal geodesic ray. If $\{\phi_m\} \subset \mathcal{H}^{\text{NA}}(L)$ is any sequence*

that converges strongly to Φ_{NA} , and Φ_m is the (maximal) geodesic ray associated to ϕ_m , then we have the convergence:

$$(3) \quad (\mathbf{E}^{dd^c\psi_Q})'^\infty(\Phi) = \lim_{m \rightarrow +\infty} (\mathbf{E}^{dd^c\psi_Q})'^\infty(\Phi_m).$$

As a consequence we always have the following identity for any maximal geodesic ray (see (40)):

$$(4) \quad (\mathbf{E}^{dd^c\psi_Q})'^\infty(\Phi) = (\mathbf{E}^{\mathcal{Q}_C})^{NA}(\Phi_{NA}).$$

REMARK 1.4. – Boucksom and Jonsson told me that the result in Theorem 1.3 is independently known to them. Indeed, our proof of this result depends on some estimates that first appeared in [5] and later refined in [4, 6].

In fact, a more general result is true (see Theorem 4.1), which will be used in our study of YTD conjecture in Section 6. The above two results, combined with the recent progresses mentioned above, essentially reduce the proof of (the \mathbb{G} -uniform version of) YTD conjecture for cscK metrics to proving that the entropy slope is algebraically approximable. More precisely it now suffices to prove the following result:

CONJECTURE 1.5. – Let Φ be a maximal geodesic ray.

Then there is a sequence $\phi_m \in \mathcal{H}^{NA}(L)$ converging to Φ_{NA} in the strong topology such that (see (19) for \mathbf{H}^{NA})

$$(5) \quad \mathbf{H}'^\infty(\Phi) \geq \lim_{m \rightarrow +\infty} \mathbf{H}^{NA}(\phi_m).$$

One difficulty in proving (5) is that there is not yet an explicit formula for \mathbf{H}'^∞ for general maximal geodesic rays. On the other hand, for any finite energy $\phi \in \mathcal{E}^{1,NA}(L)$, Boucksom-Jonsson in [18] defined the following non-Archimedean entropy in their study of K-stability (see (46)–(47)):

$$(6) \quad \mathbf{H}^{NA}(\phi) = \int_{X^{NA}} A_X(x) \mathrm{MA}^{NA}(\phi)(x).$$

Here we conjecture

CONJECTURE 1.6. – For any maximal geodesic ray Φ , $\mathbf{H}'^\infty(\Phi) = \mathbf{H}^{NA}(\Phi_{NA})$.

This is partially answered by the following results:

THEOREM 1.7. – 1. For any geodesic ray Φ , we always have the inequality:

$$(7) \quad \mathbf{H}'^\infty(\Phi) \geq \mathbf{H}^{NA}(\Phi_{NA}).$$

2. Given an ample test configuration $\pi : (\mathcal{X}, \mathcal{L}) \rightarrow \mathbb{C}$, let $\Phi = \{\varphi(s)\}$ be a geodesic ray associated to $(\mathcal{X}, \mathcal{L})$. Then we have the following slope formula:

$$(8) \quad \begin{aligned} \mathbf{H}'^\infty(\Phi) &= \mathbf{H}^{NA}(\mathcal{X}, \mathcal{L}) = K_{\tilde{\mathcal{X}}/\mathbb{P}^1}^{\log} \cdot \bar{\mathcal{L}}^n, \\ \mathbf{M}'^\infty(\Phi) &= \mathbf{M}^{NA}(\mathcal{X}, \mathcal{L}) = K_{\tilde{\mathcal{X}}/\mathbb{P}^1}^{\log} \cdot \bar{\mathcal{L}}^n + \frac{S}{n+1} \bar{\mathcal{L}}^{n+1}. \end{aligned}$$