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FOURIER DECAY, RENEWAL THEOREM AND SPECTRAL GAPS FOR RANDOM WALKS ON SPLIT SEMISIMPLE LIE GROUPS

BY JIALUN LI

ABSTRACT. – We establish an exponential error term for the renewal theorem in the context of products of random matrices, which is surprising compared with classical abelian cases. An essential tool is the Fourier decay of the Furstenberg measures on the projective spaces, which is a higher dimensional generalization of a recent work of Bourgain-Dyatlov.

RÉSUMÉ. – On établit un terme d'erreur exponentiel dans le théorème de renouvellement dans le cadre de produits de matrices aléatoires, qui est inattendu par rapport au cas classique abélien. L'outil clef est la décroissance de Fourier de mesures de Furstenberg sur les espaces projectifs, qui est une généralisation en dimension supérieure d'un travail récent de Bourgain-Dyatlov.

1. Introduction

Let V be a nontrivial finite-dimensional irreducible real algebraic representation of a $\mathbb{R}$-split semisimple algebraic real Lie group G (for example $G = \mathrm{SL}_{m+1}(\mathbb{R})$ and $V = \mathbb{R}^{m+1}$ with $m \geq 1$). Let μ be a Borel probability measure on G and let Γ_μ be the subgroup generated by the support of μ. We call μ Zariski dense if Γ_μ is a Zariski dense subgroup of G, which means that the measure μ does not concentrate on any proper algebraic subgroup of G. We also need the hypothesis of the finite exponential moment. If V is a faithful representation of G, the definition of the exponential moment is that there exists ϵ positive such that

$$\int_G \|g\|^\epsilon \mathrm{d}\mu(g) < \infty,$$

where $\|g\|$ is the operator norm of g acting on V. For the general case, please see Definition 2.50. From now on, we always suppose that the measure μ is Zariski dense with a finite exponential moment.

For any natural number n, let μ^{*n} be the n-times convolution of the measure μ. Let $X_1, \ldots, X_n$ be i.i.d. random variables in G with the same distribution μ, then μ^{*n} is the distribution of the product $X_1 X_2 \cdots X_n$. People are interested in generalizing results about

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the sum of random variables, such as the law of large numbers or the central limit theorem, to the norms or coefficients of products of random matrices. The pioneers of the study of products of random matrices are Furstenberg, Kesten, Guivarc'h. For example, Furstenberg proved the law of large numbers, which states that there exists a Lyapunov constant $\sigma_{V,\mu} > 0$ such that almost surely

$$\lim_{n\to\infty} \frac{1}{n} \log \|X_1 \cdots X_n\| = \sigma_{V,\mu}.$$

Renewal theorem

Our first result is about exponential error term in the renewal theorem. The renewal theorem was first introduced in the sum of random variables by Blackwell and in products of random matrices by Kesten [23] [24], where he applied the renewal theorem to study the solution of random difference equations. The renewal theorem also has applications outside probability theory, for example, its application to the decay of correlation by Sarig [39] and its application to the asymptotic analysis of certain counting functions arising in the geometry of discrete groups by Lalley [25].

Let $\|\cdot\|$ be a good norm (1) on V. Recall that for g in G, we define $\|g\|$ to be its operator norm on V. For a compactly supported continuous function f on $\mathbb{R}$ and a real number t, we define the renewal sum for norms by

$$R_P f(t) := \sum_{n=0}^{+\infty} \int_G f(\log \|g\| - t) \mathrm{d}\mu^{*n}(g).$$

If f is positive, this sum is obviously well defined. In fact, using the positiveness of the Lyapunov constant and the Large deviation principle, we can show that this sum is always well defined.

Let $X = \mathbb{P}V$ be the real projective space of V, which is the set of lines of V. Then we have a group action of G on X. We define the cocycle function $\sigma : G \times X \to \mathbb{R}$ by, for $x = \mathbb{R}v$ in X and g in G,

$$\sigma(g, x) = \log \frac{\|gv\|}{\|v\|}. \tag{1.1}$$

For a compactly supported continuous f on $\mathbb{R}$, the renewal sum for cocycles is defined by

$$Rf(x, t) := \sum_{n=0}^{+\infty} \int_G f(\sigma(g, x) - t) \mathrm{d}\mu^{*n}(g), \text{ for } x \in X \text{ and } t \in \mathbb{R}.$$

The limit law for norms and the limit law for cocycles are closely related.

The renewal theorem gives us a phenomenon of equidistribution when the time t is large enough. The main result (due to Guivarc'h and Le Page [19]) is that for a compactly supported continuous function f when the time t tends to infinite, the renewal sum $Rf(x, t)$ tends to $\frac{1}{\sigma_{V,\mu}} \int f$, where $\sigma_{V,\mu}$ is the Lyapunov constant.

Here is one of our main results.

(1) When $G = \mathrm{SL}_{m+1}(\mathbb{R})$ and $V = \mathbb{R}^{m+1}$, any euclidean norm on $\mathbb{R}^{m+1}$ is a good norm. For the definition, please see Definition 2.8.

THEOREM 1.1. – *Let* $\mathbf{G}$ *be a connected semisimple algebraic group defined and split over* $\mathbb{R}$ *and let* $G = \mathbf{G}(\mathbb{R})$ *be its group of real points. Let* μ *be a Zariski dense Borel probability measure on* G *with a finite exponential moment. Let* V *be a nontrivial finite-dimensional irreducible real algebraic representation of* G *with a good norm. There exists* $\epsilon > 0$ *such that for* $f \in C_c^2(\mathbb{R})$ *and* $t \in \mathbb{R}$*, we have*

$$Rf(x,t) = \frac{1}{\sigma_{V,\mu}} \int_{-t}^{\infty} f(u) \mathrm{d}Leb(u) + O_f(e^{-\epsilon|t|}),$$

and

$$R_P f(t) = \frac{1}{\sigma_{V,\mu}} \int_{-t}^{\infty} f(u) \mathrm{d}Leb(u) + O_f(e^{-\epsilon|t|}),$$

where O_f *depends on the support and some Sobolev norm of* f.

REMARK 1.2. – We should compare this result with the renewal theorem on $\mathbb{R}$ (the classical abelian case). If λ is a measure on $\mathbb{R}$ whose support is finite and generate a dense subgroup of $\mathbb{R}$, then the error term in the renewal theorem for the real random walk induced by λ is never exponential. So Theorem 1.1 says that the behavior of products of random matrices is more regular. While in lattice case, that is the support of λ is contained in a discrete additive subgroup of $\mathbb{R}$, a version of exponential speed is known in [42].

One application of Theorem 1.1 is the Fourier decay of self-affine measures in [30], where the exponential error term is crucially used to obtain a polynomial rate in the decay of Fourier transform.

Our result is an improvement of Boyer's result [12], where the error term is polynomial on t.

We hope this type of result and its idea of proof will help obtain some exponential error terms in the orbital counting problems of higher ranks. For instance in [25], [36] and [38], they are interested in the asymptotic growth of $\#\{\gamma \in \Gamma |\ d(\gamma o, o) \leq R\}$, where o is the base point in $\mathrm{SL}_{m+1}(\mathbb{R})/\mathrm{SO}(m+1)$ and Γ is a discrete subgroup of $\mathrm{SL}_{m+1}(\mathbb{R})$. In [16], they relate the number of integer solutions of Markoff-Hurwitz equations to a counting problem of discrete subsemigroup of $\mathrm{SL}_{m+1}(\mathbb{R})$, whose limit set is known as the Rauzy gasket.

This type of error term is usually proved using some spectral gap property.

Spectral gap

Equip $\mathbb{P}V$ with a Riemannian distance. For γ positive, let $C^\gamma(\mathbb{P}V)$ be the space of γ-Hölder functions on $\mathbb{P}V$. We introduce the transfer operator, which is an analogue of the characteristic function for real random variables.

DEFINITION. – *For* z *in* $\mathbb{C}$ *with the real part* $|\Re z|$ *small enough, let* P_z *be the operator on the space of continuous functions on* $\mathbb{P}V$*, which is given by*

$$P_z f(x) = \int_G e^{z\sigma(g,x)} f(gx) \mathrm{d}\mu(g), \text{ for } x \in \mathbb{P}V,$$

where the cocycle $\sigma(g,x)$ *is defined in* (1.1).