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RIGIDITY CONJECTURES FOR CONTINUOUS QUOTIENTS

BY ALESSANDRO VIGNATI

ABSTRACT. – We prove several rigidity results for corona C*-algebras and Čech-Stone remainders under the assumption of Forcing Axioms. In particular, we prove that a strong version of Todorčević's OCA and Martin's Axiom at level $\aleph_1$ imply: (i) that if X and Y are locally compact second countable topological spaces, then all homeomorphisms between $\beta X \setminus X$ and $\beta Y \setminus Y$ are induced by homeomorphisms between cocompact subspaces of X and Y; (ii) that all automorphisms of the corona algebra of a separable C*-algebra are trivial in a topological sense; (iii) that if A is a unital separable infinite-dimensional C*-algebra, the corona algebra of $A \otimes \mathcal{K}(H)$ does not embed into the Calkin algebra. All these results do not hold under the Continuum Hypothesis.

RÉSUMÉ. – Nous prouvons plusieurs résultats de rigidité pour les C*-algèbres de couronnes et les restes de Čech-Stone sous l'hypothèse des axiomes du forcing. En particulier, nous prouvons qu'une version forte de l'axiome du coloration ouvert de Todorčević et l'axiome de Martin au niveau $\aleph_1$ impliquent : (i) que si X et Y sont des espaces topologiques localement compacts et à base dénombrable, alors tous les homéomorphismes entre $\beta X \setminus X$ et $\beta Y \setminus Y$ sont induits par des homéomorphismes entre sous-espaces cocompacts de X et Y ; (ii) que tous les automorphismes des C*-algèbres de couronne séparables sont triviaux en un sens topologique ; (iii) que si A est une C*-algèbre de dimension infinie unitaire et séparable, l'algèbre couronne de $A \otimes \mathcal{K}(H)$ ne se plonge pas dans l'algèbre de Calkin. Tous ces résultats sont en défaut sous l'hypothèse du continu.

1. Introduction

This paper is squarely placed in the interface between set theory and operator algebras, with feedback into general topology. Its main goal is to study automorphisms of corona C*-algebras.

We begin by introducing the general framework. Let X and Y be two Polish spaces carrying an algebraic structure compatible with the topology. Let $I \subseteq X$ and $J \subseteq Y$ be

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two definable (e.g., Borel, or analytic) substructures inducing quotients X/I and Y/J.[1] Can we characterize the isomorphisms between the quotients X/I and Y/J in terms of the topological, or algebraic, structure of X, Y, I, and J? If X/I and Y/J are isomorphic, what are the relations between X and Y, and I and J?

Given an isomorphism $\phi\colon X/I \to Y/J$, one searches for a map $\Phi\colon X \to Y$ preserving some topological or algebraic structure, while making the following diagram commute:

$$\begin{array}{ccc} X & \xrightarrow{\quad\Phi\quad} & Y \\ \downarrow & & \downarrow \\ X/I & \xrightarrow{\quad\phi\quad} & Y/J. \end{array}$$

An isomorphism $\phi\colon X/I \to Y/J$ is

— topologically trivial if there is a lifting map $\Phi\colon X \to Y$ preserving (some of) the topological structure;

— algebraically trivial if there is a lifting map $\Phi\colon X \to Y$ preserving (some of) the algebraic structure.

The quotients X/I and Y/J are topologically (algebraically) isomorphic if there is a topologically (algebraically) trivial isomorphism $X/I \to Y/J$. While the technical definition of (topologically or algebraically) trivial depends on the category one is interested in (e.g., Boolean algebras, groups, rings, C*-algebras,...), there are two questions common throughout different categories.

Question A. – (a) Are all isomorphisms $X/I \to Y/J$ topologically trivial?

(b) Are all topologically trivial isomorphisms algebraically trivial?

Shoenfield's Absoluteness Theorem ([35, Theorem 13.15]) implies that answers to Question A(b) cannot be changed by forcing. On the contrary, the existence of isomorphisms that are not topologically trivial is in many cases subject to the axioms in play; more than that, thanks to Woodin's Σ^2_1-absoluteness ([54]), the Continuum Hypothesis[2] CH provides the optimal set-theoretic assumption for obtaining isomorphisms that are not topologically trivial. Also, for large classes of quotients it is consistent with ZFC that all isomorphisms between the objects in consideration are topologically trivial. Often, these 'rigidity' phenomena happen in presence of Forcing Axioms[3]. One of the most important among Forcing Axioms is Shelah's Proper Forcing Axiom PFA ([45]), which historically provides the environment for rigidity results, although there is no metatheorem showing that PFA (or some of its consequences) is the optimal environment for the statement 'all isomorphisms are topologically trivial' to hold. We will prove our rigidity results using two Forcing Axioms which are consequences of PFA, the axioms OCA and $\mathrm{MA}_{\aleph_1}$. Both axioms

[1] An operator algebraist should think at $X = \mathcal{M}(A)$ and $Y = \mathcal{M}(B)$, the multiplier algebras of two nonunital C*-algebras $I = A$ and $J = B$.

[2] CH is the statement $2^{\aleph_0} = \aleph_1$, the latter being the first uncountable cardinal.

[3] Forcing Axioms can be viewed as higher versions of Baire's category theorem, see §2.1.

are in contradiction with CH; differently from PFA whose consistency needs large cardinals, the consistency of OCA + $\mathrm{MA}_{\aleph_1}$ can be proved within ZFC ([49, §2]).

The first concrete case of study was the one of quotients of the Boolean algebra $\mathcal{P}(\mathbb{N})$, considered with its natural Polish topology. An isomorphism between quotients $\mathcal{P}(\mathbb{N})/\mathscr{I}$ and $\mathcal{P}(\mathbb{N})/\mathscr{J}$, for definable ideals $\mathscr{I}, \mathscr{J} \subseteq \mathcal{P}(\mathbb{N})$, is algebraically trivial if it is induced by a Boolean algebra endomomorphism of $\mathcal{P}(\mathbb{N})$, and topologically trivial if it admits a Baire-measurable lifting, without the requirement of preserving any algebraic operation. Thanks to Ulam stability phenomena for finite Boolean algebras, see [15, §1.8], if $\mathscr{I}$ is the ideal of finite sets Fin, the notions of topologically and algebraically trivial automorphisms coincide, and we will simply refer to trivial automorphisms. (This is not yet proved to be the case for all quotients of $\mathcal{P}(\mathbb{N})$ by a Borel ideal. For more on this, see [13].)

The question on whether all automorphisms of $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ are trivial arose from the study of the homogeneity properties of the space $\beta\mathbb{N} \setminus \mathbb{N}$. Rudin ([43]) showed that CH implies the existence of nontrivial automorphisms of $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$. Also, Shelah proved in [45] that the assertion 'all automorphisms of $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ are trivial' is consistent with ZFC. This was later shown to be implied by PFA in [46]. This assumption was weakened to that of OCA and $\mathrm{MA}_{\aleph_1}$ in [50]. Extending the focus to quotients by more general Borel ideals, the eye-opening work of Farah [15] showed that in presence again of OCA and $\mathrm{MA}_{\aleph_1}$, in many cases [(4)], all isomorphisms between quotients $\mathcal{P}(\mathbb{N})/\mathscr{I}$ and $\mathcal{P}(\mathbb{N})/\mathscr{J}$ are algebraically trivial. In this case, the quotients are isomorphic if and only if $\mathscr{I}$ and $\mathscr{J}$ are isomorphic themselves. This fails in many situations in presence of CH: in [33] it was showed that all quotients over F_σ ideals are countably saturated. This, together with a Cantor's back-and-forth argument, shows that under CH all quotients over F_σ ideals are pairwise isomorphic, and additionally each such quotient has $2^{2^{\aleph_0}}$ distinct automorphisms that are not topologically trivial. (For more information on the situation in the Boolean algebra setting, see [25, 17])

Such a general approach was undertaken in the study of other quotient structures such as groups and semilattices, sometimes with limited success ([16]). The next category where this way of thinking proved to be successful is the one of C*-algebras, the focus being particular quotients known as corona algebras, the forefather being the Calkin algebra $\mathcal{Q}(H)$. Since abelian C*-algebras correspond, by a controvariant equivalence of categories, to locally compact topological spaces, the study of C*-algebras is often viewed as 'noncommutative topology'. Following this philosophy, one sees $\mathcal{B}(H)$, the algebra of bounded linear operators on a separable complex Hilbert space H, as the noncommutative analog of the algebra $\ell_\infty(\mathbb{N})$, and $\mathcal{K}(H)$, the ideal of compact operators on H, as the noncommutative $c_0(\mathbb{N})$. The Calkin algebra $\mathcal{Q}(H) = \mathcal{B}(H)/\mathcal{K}(H)$ is then thought as the noncommutative analog of $\ell_\infty(\mathbb{N})/c_0(\mathbb{N})$. Note that the Boolean algebras of projections in $\ell_\infty(\mathbb{N}), c_0(\mathbb{N})$, and $\ell_\infty(\mathbb{N})/c_0(\mathbb{N})$ are, respectively, isomorphic to $\mathcal{P}(\mathbb{N})$, Fin, and $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$. (For the many parallels between $\mathcal{Q}(H)$ and $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ see [22, 23, 53, 55].)

The interest in the study of automorphisms of $\mathcal{Q}(H)$ has operator algebraic roots: while developing the connections between algebraic topology, index theory, and operator algebras, in introducing extensions and K-homology for C*-algebras, Brown, Douglas, and Fillmore asked in [5] whether there can exist an outer automorphism of $\mathcal{Q}(H)$. Their main

(4) Assumptions on $\mathscr{I}$ or $\mathscr{J}$ should be given here, see [15, §3.4].