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AROUND BRODY LEMMA

by

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Abstract. – Brody's lemma is a basic tool in hyperbolicity. It provides an entire curve, i.e., a non constant holomorphic image of the complex line, out of a diverging sequence of holomorphic discs. Consequently Brody's Lemma characterizes hyperbolicity in terms of absence of entire curves.

We present direct applications of Brody's Lemma, including the Green theorem (hyperbolicity of the complement of 5 lines in the projective plane) and an example of a hyperbolic surface of degree 6 in the projective space. We also describe a variant of Brody's lemma aiming to better localize the entire curve it produces.

As a byproduct of this variant, hyperbolicity is characterized in terms of a linear isoperimetric inequality for holomorphic discs.

1. Brody lemma

Let X be a compact complex manifold. An *entire curve* in X is a non constant holomorphic map $f : \mathbb{C} \to X$. It is a *Brody curve* if its derivative $\|f'\|$ is bounded, where the norm is computed with respect to the standard metric on $\mathbb{C}$ and a given riemannian metric on X. Brody curves arise naturally as limits of sequences of larger and larger holomorphic disks, thanks to Brody lemma [**5**].

Brody lemma. – *Let $f_n : \mathbb{D} \to X$ a sequence of holomorphic maps from the unit disk to a compact complex manifold. Suppose $\|f_n'(0)\|$ unbounded. Then there exist affine reparametrizations r_n of $\mathbb{C}$ such that $f_n \circ r_n$ converges locally uniformly toward a Brody curve, after extracting a subsequence.*

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Proof. – We may suppose f_n smooth up to the boundary (replacing $f_n(z)$ by $f_n(\frac{z}{2})$). Denote by $\delta(z)$ the euclidean distance from z to $\partial\mathbb{D}$. As the function $\delta\|f_n'\|$ vanishes on $\partial\mathbb{D}$ it reaches its maximum inside $\mathbb{D}$, say at a_n. This is where we will reparametrize f_n. Let D_n be the disk $D(a_n, \frac{\delta(a_n)}{2})$. We have $\|\delta f_n'\|_{D_n} \leq \delta(a_n)\|f_n'(a_n)\|$. But $\delta \geq \frac{\delta(a_n)}{2}$ on D_n so $\|f_n'\|_{D_n} \leq 2\|f_n'(a_n)\|$.

Define the reparametrization by $r_n(z) = a_n + \frac{z}{\|f_n'(a_n)\|}$. let D_n' be the preimage of D_n by r_n. Its radius is unbounded as $\delta(a_n)\|f_n'(a_n)\| \geq \|f_n'(0)\|$. We may suppose that D_n' increases toward $\mathbb{C}$ after extracting. Moreover $\|(f_n \circ r_n)'\|_{D_n'} \leq 2$ and $\|(f_n \circ r_n)'(0)\| = 1$. By Ascoli theorem we may extract a subsequence of $f_n \circ r_n$ which converges locally uniformly toward a holomorphic map $f : \mathbb{C} \to X$ such that $\|f'\|_{\mathbb{C}} \leq 2$ and $\|f'(0)\| = 1$. It is a Brody curve. □

As a consequence we get a characterization of Kobayashi-hyperbolicity.

Recall that the *Kobayashi pseudometric* of X at p in the direction v is $K(p,v) = \inf\{r > 0 \mid \exists f : \mathbb{D} \to X \text{ holomorphic}, f(0) = p, f'(0) = \frac{v}{r}\}$. It measures the size of the holomorphic disks passing through a point in a given direction (the larger the disk through p in direction v the smaller $K(p,v)$). The manifold X is *Kobayashi-hyperbolic* if its pseudometric is non degenerate.

Criterion. – *A compact complex manifold X is Kobayashi-hyperbolic if and only if there is no entire curve in it.*

Indeed the vanishing of $K(p,v)$ gives rise to a sequence of holomorphic disks $f_n : \mathbb{D} \to X$ such that $f_n(0) = p$ and $f_n'(0)$ is unbounded in the direction of v. By Brody lemma we get a Brody curve in X. Conversely the Kobayashi pseudometric has to vanish along any entire curve.

In the sequel we will say that $U \subset X$ is *hyperbolic* if U does not contain any entire curve. For instance $\mathbb{D}$ is hyperbolic by Liouville theorem. Hyperbolicity is invariant under étale covering. Indeed if $U \to V$ is such a covering, any entire curve in U may be pushed down in an entire curve in V. Conversely any entire curve in V can be lifted to an entire curve in U. For instance compact complex curves of genus ≥ 2 are hyperbolic as they are uniformized by $\mathbb{D}$. Another consequence of Brody lemma is that hyperbolicity is an open property.

Openness. – *Let X be a compact complex manifold and $F \subset X$ a closed subset. If F is hyperbolic then so is any sufficiently small neighborhood of F.*

Otherwise we would get an entire curve in each ϵ-neighborhood F_ϵ of F. This would allow us to construct a sequence of holomorphic disks $f_n : \mathbb{D} \to F_{\frac{1}{n}}$ such that $\|f_n'(0)\|$ is unbounded, giving at the limit a Brody curve in F.

We discuss now some examples of hyperbolic complex surfaces. We start with the simplest hyperbolic complement, generalizing Picard theorem (the hyperbolicity of $\mathbb{P}^1(\mathbb{C})\backslash 3$ points).

Green theorem **[9]**. – *Let L be a collection of five lines in general position in $\mathbb{P}^2(\mathbb{C})$. Then $\mathbb{P}^2(\mathbb{C}) \setminus L$ is hyperbolic.*

Here general position means there is no triple point in the configuration.

Proof. – Embedd $\mathbb{P}^2(\mathbb{C})$ into $\mathbb{P}^4(\mathbb{C})$ by $z \mapsto [l_1(z) : \cdots : l_5(z)]$ using the equations of the lines. Call P its image and $P^* = P \setminus H$ the complement of the collection H of coordinate hyperplanes in $\mathbb{P}^4(\mathbb{C})$. We want to prove the hyperbolicity of P^*. Let F_n be the self-map of $\mathbb{P}^4(\mathbb{C})$ given by $z \mapsto [z_1^n : \cdots : z_5^n]$. It induces an étale covering from $\mathbb{P}^4(\mathbb{C}) \setminus H$ to itself, so the hyperbolicity of P^* and $F_n^{-1}(P^*)$ are equivalent. The point is that $F_n^{-1}(P)$ converges toward a polyhedron whose hyperbolicity is easily checked by Liouville theorem. This will conclude the proof by openness.

Let us make it precise.

By the general position assumption, P avoids the coordinate lines $(z_i = z_j = z_k = 0)$. So P is contained in $X_\epsilon = \bigcap_{\{i,j,k\}}(\max(|z_i|, |z_j|, |z_k|) \geq \epsilon\|z\|)$ for some small $\epsilon > 0$. Here $\|z\| = \max|z_i|$. Now $F_n^{-1}(X_\epsilon) = X_{\epsilon^{\frac{1}{n}}}$ decreases toward the polyhedron $X = X_1$ when n goes to infinity. Note that on X $\|z\|$ is reached on one out of three arbitrary components of z, meaning that $\|z\|$ is always reached on three components at least. So X is alternatively seen as a finite union of faces $X_{i,j,k} = (|z_i| = |z_j| = |z_k| = \|z\|)$.

Let us check its hyperbolicity. Let $f : \mathbb{C} \to \mathbb{P}^4(\mathbb{C})$ be a holomorphic map. It lifts to $(\mathbb{C}^5)^*$ (essentially because $H^1(\mathbb{C}, \mathcal{O}^*) = 0$, see **[10]**) so $f(z) = [f_1(z) : \cdots : f_5(z)]$ where f_i is holomorphic. Now if $f(\mathbb{C}) \subset X$ it has to spend some time in one of the faces, say $X_{1,2,3}$. This implies by analytic continuation that $|f_1| = |f_2| = |f_3|$ everywhere. But $\|f\| = \max(|f_1|, |f_2|, |f_3|)$ by definition of X. Hence $|f_1|$ dominates the other components everywhere, meaning that f is bounded in the chart $(z_1 = 1)$ thus constant by Liouville theorem. Therefore X does not contain any entire map. □

Remark. – This argument is dynamical in essence as X is nothing but an intermediate Julia set for F_2 which is known to attract backward iterates of a generic plane. It stems from the proof of Picard theorem by A. Ros **[17]** and works in any dimension **[2]**.

We focus now on examples of hyperbolic surfaces of low degree in $P^3(\mathbb{C})$. This fits into the framework of Kobayashi conjecture which predicts that a generic surface of degree ≥ 5 in $\mathbb{P}^3(\mathbb{C})$ is hyperbolic. It holds true for degree ≥ 18 **[16]** but few examples of hyperbolic surfaces of smaller degree are known, according to the motto "the lower the degree the harder the hyperbolicity". Here we adapt a deformation method due to M. Zaidenberg **[19]** to produce examples of degree 6 by reduction to Green theorem. This also works for higher degrees and higher dimensions **[11]**. Note that the following remains open.

Question. – *Find a hyperbolic quintic surface in $\mathbb{P}^3(\mathbb{C})$.*

We will use Brody lemma in the following form.

Sequences of entire curves. – *Let X be a compact complex manifold. Then any sequence of entire curves in X can be made converging toward an entire curve after reparametrization and extraction.*

Indeed let (f_n) be the sequence of entire curves. By translating we may suppose that $f_n'(0)$ does not vanish and by dilating that actually $\|f_n'(0)\|$ is unbounded. It remains to apply Brody lemma.

We will also invoke the following fact.

Stability of intersections. – *Let X be a complex manifold and $H \subset X$ an analytic hypersurface. Suppose that a sequence (f_n) of entire curves in X converges toward an entire curve f. If $f(\mathbb{C})$ is not contained in H then $f(\mathbb{C}) \cap H \subset \lim f_n(\mathbb{C}) \cap H$.*

Indeed let z be a point in $f^{-1}(H)$ and h a local equation of H near $f(z)$. As $h \circ f$ does not vanish identically near z, $h \circ f_n$ has to have a zero in any small neighborhood of z for large n by Rouché theorem.

We construct now our example of hyperbolic sextic surface as a suitable small deformation of a union of six planes (see also **[6]**, **[7]** for other examples).

A hyperbolic sextic surface. – *Let $(P_i = (p_i = 0))$ be a collection of six planes in general position in $P^3(\mathbb{C})$. Then we can find a sextic surface $S = (s = 0)$ such that the surface $\Sigma_\epsilon = (\Pi p_i = \epsilon s)$ is hyperbolic for $\epsilon \neq 0$ sufficiently small.*

Here general position means there is no quadruple point in the configuration. Moreover S will be in general position with respect to the P_i, in the sense that it will avoid the triple points of the configuration. Note that by definition $\Sigma_\epsilon \cap P_i \subset S$.

Proof. – The first step reduces the problem to the hyperbolicity of complements. It is the heart of Zaidenberg's method (see **[19]**) and goes as follows. If Σ_{ϵ_n} is not hyperbolic for ϵ_n going to zero we have entire curves $f_n : \mathbb{C} \to \Sigma_{\epsilon_n}$. By Brody lemma we get at the limit an entire curve $f : \mathbb{C} \to \Sigma_0 = \cup P_i$. It lands inside one of the planes.

We analyze now its position with respect to the other planes. The crucial remark is the following. If $f(\mathbb{C})$ is not contained in P_i then $f(\mathbb{C}) \cap P_i \subset S$. Indeed by stability of intersections

$$f(\mathbb{C}) \cap P_i \subset \lim f_n(\mathbb{C}) \cap P_i \subset \lim \Sigma_{\epsilon_n} \cap P_i \subset S.$$

We infer that $f(\mathbb{C})$ cannot land into a double line of the configuration of planes. If it were the case $f(\mathbb{C})$ would have to avoid the 4 triple points on the line by the remark and the general position of S, contradicting Picard theorem.

We end up with $f(\mathbb{C})$ contained in one plane and avoiding the others except at points of S, again by the remark. So $f(\mathbb{C})$ is in a complement of the form $P_i \setminus (\bigcup_{j \neq i} P_j \setminus S)$. Hence we are finished if we are able to find a sextic surface S such that all these complements are hyperbolic.

The second step consists in constructing this sextic surface. Note that the situation is similar to Green theorem. We have plane complements of five lines on which a few points are deleted. To create S we proceed by deformation in order to remove these points on more and more double lines. Our starting point is the collection of complements $P_i \setminus (\bigcup_{j\neq i} P_j)$ which are hyperbolic by Green theorem.

We want to remove points on a double line, say $L = P_1 \cap P_2$, keeping the hyperbolicity. For this consider a sextic surface $S_0 = (s_0 = 0)$ in general position with respect to the P_i and deform it toward the union of the remaining planes by taking $S_1 = (p_3p_4p_5^2p_6^2 = \epsilon_0 s_0)$ for a small $\epsilon_0 \neq 0$. In restriction to L, this pushes the points of the sextic surface toward the triple points. So the complement $P_1 \setminus (\bigcup_{i\neq 1} P_i \setminus (L \cap S_1))$ is close to $P_1 \setminus (\bigcup_{i\neq 1} P_i)$, hence hyperbolic by a suitable openness argument.

This is a particular case of the following lemma which we apply inductively to conclude. □

Lemma. – *Let Δ_k be a collection of k double lines, $L = P_{i_1} \cap P_{i_2}$ an extra one and $\Delta_{k+1} = \Delta_k \cup L$. Assume $S_k = (s_k = 0)$ already constructed such that the complements $P_i \setminus (\bigcup_{j\neq i} P_j \setminus (\Delta_k \cap S_k))$ are hyperbolic. Then so are $P_i \setminus (\bigcup_{j\neq i} P_j \setminus (\Delta_{k+1} \cap S_{k+1}))$ where $S_{k+1} = (p_{i_3}p_{i_4}p_{i_5}^2p_{i_6}^2 = \epsilon_k s_k)$ for any small enough $\epsilon_k \neq 0$.*

Note that S_{k+1} is still in general position with respect to the P_i if S_k was. Remark also that the geometry does not change on Δ_k. We have $\Delta_k \cap S_k = \Delta_k \cap S_{k+1}$.

Proof of the lemma. – Take $L = P_1 \cap P_2$ for simplicity. If we cannot find such an ϵ_k, we have a sequence of sextic surfaces $S_{k+1,n}$ converging toward $P_3 \cup P_4 \cup P_5 \cup P_6$ and entire curves $f_n(\mathbb{C})$ sitting in one of the corresponding complements, say $P_1 \setminus (\bigcup_{j\geq 2} P_j \setminus ((\Delta_k \cap S_k) \cup (L \cap S_{k+1,n})))$. We get at the limit an entire curve $f(\mathbb{C})$ in P_1. As before it cannot degenerate inside a double line. By stability of intersections, for $j \geq 2$ we have $f(\mathbb{C}) \cap P_j \subset \lim f_n(\mathbb{C}) \cap P_j \subset (\Delta_k \cap S_k) \cup \lim L \cap S_{k+1,n}$. If $j \geq 3$ we infer that $f(\mathbb{C}) \cap P_j \subset \Delta_k \cap S_k$ as $P_j \cap L \cap S_{k+1,n}$ is empty by general position. Note now that $\lim\ L \cap S_{k+1,n}$ consists in the triple points of L hence sits in $\bigcup_{j\geq 3} P_j$. Then thanks to the previous case we also have $f\mathbb{C}) \cap P_2 \subset \Delta_k \cap S_k$. Therefore $f(\mathbb{C})$ lands in $P_1 \setminus (\bigcup_{j\geq 2} P_j \setminus (\Delta_k \cap S_k))$ contradicting the hypothesis. □

2. A variant

A drawback of Brody lemma is the lack of information about the location of the entire curve it produces. It might land far away from the points where the disks blow up. Here is a simple example due to J. Winkelmann (see also **[18]**, and **[10]** for background on blow-ups).

Example. – Let $A = \mathbb{C}^2/(\mathbb{Z} \oplus i\mathbb{Z})^2$ be the standard torus and $\pi : \tilde{A} \to A$ the blow-up of A at a point p. Take a dense injective line L in A, say $L = (z_2 = \lambda z_1)$ for λ irrational. Consider the sequence of disks $f_n(\mathbb{D})$ on L given by $f_n(z) = (nz, \lambda nz)$. Let $\tilde{f}_n$ the strict transform of f_n. If $\tilde{f}$ is a Brody curve obtained from the $\tilde{f}_n$ by reparametrization, as in the Brody lemma, then $\tilde{f}(\mathbb{C})$ is contained in the exceptional divisor E.