Panoramas & Synthèses
56, 2021, p. 209–252

KOBAYASHI HYPERBOLICITY, NEGATIVITY OF THE CURVATURE AND POSITIVITY OF THE CANONICAL BUNDLE

by

Simone Diverio

Abstract. – We give a detailed account of a recent breakthrough by Wu and Yau, generalized shortly afterwards by Tosatti and Yang (and also by Diverio and Trapani). The breakthrough sits at the crossroad of complex differential geometry and Kobayashi hyperbolicity.

More specifically, an old conjecture of Kobayashi, stated at the very beginning of the theory, predicts that a complex projective (or more generally compact Kähler) Kobayashi hyperbolic manifold should have ample canonical bundle. On the one hand it is also known since the beginning of the theory that a compact complex manifold with a Hermitian metric whose holomorphic sectional curvature is negative is Kobayashi hyperbolic. On the other hand a compact Kähler manifold with ample canonical bundle is known—by the celebrated work of Aubin and Yau—to admit a Kähler metric with (constant) negative Ricci curvature.

Wu and Yau's theorem states that if a smooth projective manifold admits a Kähler metric with negative holomorphic sectional curvature, then it also admits a possibly different Kähler metric whose Ricci curvature is negative. The result can be therefore seen as a weak confirmation of Kobayashi's conjecture above, since it gives the same conclusion but with the stronger hypothesis about the holomorphic sectional curvature.

Beside a careful, fully detailed presentation of the proof of the Wu-Yau theorem, we take the opportunity to give some basic background material on complex differential geometry and several results, positive and negative, about the link between curvature and Kobayashi hyperbolicity. Some natural open questions are also discussed.

The proof of the Wu-Yau theorem presented here closely follows the original main ideas by Wu and Yau, but the conclusion of the proof is simplified somewhat by using the pluripotential approach of Diverio and Trapani.

1. Introduction

Let X be a compact complex manifold. An *entire curve* traced in X is by definition a non constant holomorphic map $f\colon \mathbb{C} \to X$. By Brody's criterion X is Kobayashi hyperbolic if and only if X does not admit any entire curve.

At the very beginning of the theory, in the early 70s, very few examples of (higher dimensional) compact complex manifolds where known: mainly compact quotients of bounded domains in $\mathbb{C}^n$. Such quotients admits the Bergman metric, whose class lies by construction in the opposite of the first Chern class of X. Thus, for them, the canonical bundle is positive and therefore ample. In particular such quotients are projective.

We can guess that this lack of knowledge of examples on the one hand, and the positivity property of the canonical bundle of the known hyperbolic compact complex manifolds on the other hand, led S. Kobayashi to conjecture the following.

***Conjecture* (Kobayashi '70)**. – *Let X be a compact Kähler (or projective) manifold which is Kobayashi hyperbolic. Then, K_X is ample.*

In the same vein, Kobayashi also asked in [**23**] whether a compact hyperbolic complex manifold has always infinite fundamental group. While the answer to this latter question is nowadays known to be negative (we know plenty of examples of simply connected compact complex manifold, mainly given by smooth projective general complete intersections of high degree), the former question is still widely open (and believed to be true).

Observe that, at least in the projective case, we now know since Mori's breakthrough [**29**] that being hyperbolic implies the nefness of the canonical bundle, due to the absence of rational curves. Thus, the canonical class is at least in the closure of the ample cone. What we want is then to show that hyperbolicity pushes the canonical class a little bit further into the ample cone.

Beside the class of smooth compact quotients of bounded domain, another remarkable class of compact hyperbolic manifolds—known since the beginning of the theory—is given by compact Hermitian manifolds whose holomorphic sectional curvature is negative. Even if this is for sure an important class where to test conjectures on hyperbolic manifolds, it is somehow surprising that until very recently the Kobayashi conjecture was not known even for this class.

The principal aim of this chapter is to present in full detail a proof of the following statement due to Wu and Yau, which settles Kobayashi's conjecture for negatively curved hyperbolic projective manifolds.

***Theorem 1.1* ([37])**. – *Let X be a smooth projective manifold, and suppose that X carries a Kähler metric ω whose holomorphic sectional curvature is everywhere negative. Then, X posseses a (possibly different) Kähler metric ω' whose Ricci curvature is everywhere negative. In particular, K_X is ample.*

Observe that, since the holomorphic sectional curvature decreases when the metric is restricted to smooth submanifolds, as a direct consequence one obtains that every smooth submanifold of a compact Kähler manifold with negative holomorphic sectional curvature has ample canonical bundle. This observation goes in the direction of the celebrated Lang conjecture which predicts the following.

Conjecture 1.2 **([26])**. – *Let X be a smooth projective complex manifold. Then, X is Kobayashi hyperbolic if and only if X as well as all of its subvarieties are of general type.*

Thus, since a projective manifold with ample canonical bundle is of general type, Wu and Yau's theorem is also a confirmation in the negatively curved case, as long as only smooth subvarieties are concerned, of (one direction of) Lang's conjecture. It is therefore of primordial importance to extend their result in the singular case. The exact statement one should try to prove is the following. Let X be an irreducible projective variety and suppose to be able to embed X into a projective manifold Y supporting a Kähler metric whose holomorphic sectional curvature is negative, at least locally around X. Thus, there should exists a modification [(1)] $\mu\colon \tilde{X} \to X$, with $\tilde{X}$ smooth, such that the canonical bundle $K_{\tilde{X}}$ is big.

Addendum. – The Wu-Yau-Tosatti-Yang theorem (as well as its generalization by Diverio and Trapani) has been very recently proved by H. Guenancia in [**16**] also for singular subvarieties as mentioned here above. His very general result stems from a highly non trivial generalization of ideas explained in this chapter, involving also deep results from the Minimal Model Program. Its most general version can be stated as follows.

Theorem 1.3 **(Guenancia [16**, Theorem B**])**. – *Let (X,D) be a pair consisting of a projective manifold X and a reduced divisor $D=\sum_{i\in I} D_i$ with simple normal crossings. Let ω be a Kähler metric on $X^\circ := X\setminus D$ such that there exists $\kappa_0>0$ satisfying*

$$\forall (x,v)\in X^\circ \times T_{X,x}\setminus\{0\},\quad \mathrm{HSC}_\omega(x,[v]) < -\kappa_0.$$

Then, the pair (X,D) is of log general type, that is, K_X+D is big.

If additionally ω is assumed to be bounded near D, then K_X is itself big.

This theorem thus provides a full confirmation of Lang's conjecture for compact Kähler manifolds with negative holomorphic sectional curvature.

Finally, Lang's conjecture has been settled very recently also in the particular case of compact free quotients of bounded domains (and in a slightly more general context, indeed) in [**3**].

---

(1) By definition, a modification $\mu\colon \tilde{X}\to X$ is a proper surjective holomorphic map, such that there exists a proper analytic subset $S\subset X$ with the property that $\mu|_{\tilde{X}\setminus\mu^{-1}(S)}\colon \tilde{X}\setminus\mu^{-1}(S)\to X\setminus S$ is a biholomorphism.

1.1. Organization of the chapter. – Beside the introduction, this chapter is made up of five sections.

Section 1 is devoted to build the proper background in complex differential geometry in order to go into the proof of Wu-Yau's theorem, as well as the basic notions of complex hyperbolicity. In particular we summarize the different kinds of curvature in the Riemannian setting as well as in the Hermitian setting, putting in evidence the relations of between these notions in the Kähler case. Moreover, we explain if and how the sign of a particular notion of curvature propagates to others. We also take the opportunity to recall how the negativity of the holomorphic sectional curvature gives the Kobayashi hyperbolicity of a compact Hermitian manifold.

Section 2 has a birational geometric flavor, and we try to motivate in this framework Kobayashi's conjecture as well as Wu-Yau's theorem and its generalizations using standard tools and conjectures. Namely, assuming the abundance conjecture and using the Iitaka fibration, we try to make clear how compact projective manifolds with trivial real first Chern class enter naturally into the picture and how to rule them out using the negativity (or even the quasi-negativity) of the holomorphic sectional curvature.

In Section 3 we present in full details an algebraic criterion due to J.-P. Demailly which a compact Hermitian manifold must satisfy in order to have negative holomorphic sectional curvature. As a consequence, we construct (still following Demailly) an example of smooth projective surface which is hyperbolic, has ample canonical bundle, but nevertheless does not admit any negatively curved Hermitian metric. This shows that Wu-Yau's theorem is unfortunately a confirmation of Kobayashi's conjecture only in a particular (although important) case.

Section 4 is the heart of the chapter and we present therein a complete, detailed proof of Wu-Yau's theorem. The proof is divided into several steps, in order to make the strategy more insightful. We tried to really work out every computation and estimate, perhaps even paying the price to be slightly redundant, to keep the chapter fully self-contained.

Finally, in Section 5 we present a couple of generalizations of Wu-Yau's theorem, namely the Kähler case due to V. Tosatti and X. Yang, and the (from a certain point of view, sharp) quasi-negative case due to S. Trapani and the author.

Acknowledgements. – The author wishes to express his gratitude to the anonymous referee for having read with great attention the present chapter, and for the uncountable valuables suggestions which really improved the exposition.

2. Complex differential geometric background and hyperbolicity

The material in this section is somehow standard, but we take the opportunity here to fix notations and explain some remarkable facts which are not necessarily in everybody's background. We refer to **[8, 21, 40]** for an excellent and more systematic treatment of the subject.

Let X be a complex manifold of complex dimension n, and let h be a Hermitian metric on its tangent space T_X, which is considered as a complex vector bundle endowed with the standard complex structure J inherited from the holomorphic coordinates on X. Then, the real part g of $h = g - i\omega$ defines a Riemannian metric on the underlying real manifold, while its imaginary part ω defines a 2-form on X.

Now, one can consider both the Riemannian or the Hermitian theory on X. On the one hand we have the existence of a unique connection ∇ on $T_X^{\mathbb{R}}$—the Levi-Civita connection—which is both compatible with the metric g and without torsion. Here the superscript $\mathbb{R}$ is put on T_X to emphasize that we are looking at the real underlying manifold. We call the square of this connection $R = \nabla^2$, the Riemannian curvature of $(T_X^{\mathbb{R}}, g)$. It is a 2-form with values in the endomorphisms of $T_X^{\mathbb{R}}$.

On the other hand, we can complexify T_X and decompose it as a direct sum of the eigenbundles for the complexified complex structure $J \otimes \mathrm{Id}_{\mathbb{C}}$ relatives to the eigenvalues $\pm i$:

$$T_X^{\mathbb{C}} = T_X \otimes \mathbb{C} \simeq T_X^{1,0} \oplus T_X^{0,1}.$$

We have a natural vector bundle isomorphism

$$\begin{aligned} \xi \colon T_X^{\mathbb{R}} &\to T_X^{1,0} \\ v &\mapsto \frac{1}{2}(v - iJv), \end{aligned}$$

which is moreover $\mathbb{C}$-linear: $\xi \circ J = i\xi$. There is a natural way to define a Hermitian metric on $T_X^{\mathbb{C}}$, as follows. We first consider the $\mathbb{C}$-bilinear extension $g^{\mathbb{C}}$ of g, and then its sesquilinear form $\tilde{h}$ made up using complex conjugation in $T_X^{\mathbb{C}}$:

$$\tilde{h}(\bullet, \bullet) := g^{\mathbb{C}}(\bullet, \bar{\bullet}).$$

Such a Hermitian metric realizes the direct sum decomposition above as an orthogonal decomposition. The complexification of ω, which we still call ω by an abuse of notation, is then a real positive $(1,1)$-form. These three notions, namely a Hermitian metric on T_X, a Hermitian metric on $T_X^{1,0}$, and a real positive $(1,1)$-form are essentially the same, since there is a canonical way to pass from one to the other.

Now, we know that there exists a unique connection D on $T_X^{1,0}$ which is both compatible with $\tilde{h}$ and the complex structure: the Chern connection. We call the square of this connection $\Theta = D^2$, the Chern curvature of $(T_X^{1,0}, \tilde{h})$. It is a $(1,1)$-form with values in the anti-Hermitian endomorphisms of $(T_X^{1,0}, \tilde{h})$.

A basic question is then: can we compare these two theories via ξ? The answer is classical and surprisingly simple. The Riemannian theory and the Hermitian one are the same if and only if the metric h is Kähler, i.e., if and only if $d\omega = 0$. In other words, the metric is Kähler if and only if

$$D = \xi \circ \nabla \circ \xi^{-1},$$

and of course, in this case, $\Theta = \xi \circ R \circ \xi^{-1}$ (see e.g., [**21**, §4.A]).