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# FIBRATIONS IN ALGEBRAIC GEOMETRY AND APPLICATIONS

*by*

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***Abstract.*** – We give a survey of various methods for constructing rational fibrations on algebraic varieties (i.e., dominant rational mappings of normal varieties that induce an algebraically closed extension of function fields), and their applications.

These fibrations are a major tool in the classification theory of algebraic varieties. The most important among them are the Iitaka fibration, the MRC fibration, and the Gamma fibration. We present them together with several concrete modes of use.

We discuss finally the core fibration, introduced more recently by Campana, which is a conjectural bridge between these algebraically defined fibrations and hyperbolicity.

## 0. Introduction

This paper is devoted to a crucial tool for the study of algebraic varieties, namely fibrations. We will in this survey paper use the following definition.

***Definition 0.1.*** – *A (rational) fibration on a projective variety or compact complex manifold $X$ is a dominant (rational or meromorphic) map $X \dashrightarrow Y$ with connected general fiber.*

The obvious interest of a fibration is that it allows to deduce properties of the total space from properties of the base and of the fibers. For example, the following facts hold:

1. If the base is Brody (resp. algebraically) hyperbolic, and all the fibers are Brody (resp. algebraically) hyperbolic then the total space is Brody hyperbolic.
2. If the base is of general type and the fiber is of general type, then the total space is of general type (see [**32**], [**49**]).
3. If the base is rationally connected and the fiber is rationally connected, so is the total space (see Section 3.2, [**28**]).

Note however that a general fibration, especially rational fibration, may not be very useful : Indeed, starting from any smooth projective variety $X$, and choosing a Lefschetz pencil of high degree hypersurfaces in $X$, we get a rational map $\widetilde{X} \dashrightarrow \mathbb{P}^1$ which does not say much about $X$ since the base has Kodaira dimension $-\infty$ and the fiber is of general type. What happens in this case is the fact that the blown-up locus is of codimension 2 in $X$ but of codimension 1 and ample in the fibers. The case of a morphism is much better but an interesting morphism to a smaller dimensional basis does not always exist and many interesting fibrations are only defined as rational maps. This paper will rather describe carefully constructed fibrations for which the fibers are instead simpler than the total space, reducing in principle the study to phenomena on the base. There are two ways of constructing fibrations: The first way, which is more classical in birational geometry, consists in exploiting the presence of sections of the adequate tensors, or divisors. This method was initiated by Iitaka for sections of line bundles, and the exploitation of holomorphic contravariant tensors has been developed over the time by Castenuovo-de Franchis, Bogomolov, Catanese, Campana. We will describe this method in Section 1.

The second way appears in [**1**], [**13**], [**35**] and consists in constructing geometrically the fiber through a general point $x$ by imposing that it contains all points that can be reached from $x$ by composing a certain number of allowed geometric processes. The two striking instances of this approach are the MRC fibration, where the fiber through a very general point contains all rational curves passing through this point, and the Shafarevich map or $\Gamma$-reduction, for which the fiber through a very general point contains all the varieties passing through this point and having a fundamental group with small image in the ambient space. This is described in Sections 3.2, 3.3.

A third method presented in [**16**] beautifully combines both approaches to produce the so-called core fibration which will be described in Section 4. This fibration into special varieties is conjecturally the one which allows to understand the degeneracy of the Kobayashi pseudodistance at the general point of a variety. For a long period, standard conjectures about the Kobayashi pseudodistance ([**37**], [**33**]) were the following:

***Conjecture 0.2***. – *A variety of general type has its Kobayashi pseudometric nondegenerate at a general point.*

Very nice recent evidences for this conjecture have been obtained by Diverio-Merker-Rousseau [**24**] and Brotbek [**11**], who work in the case of hypersurfaces in projective space, and by Demailly [**23**].

***Conjecture 0.3***. – *A variety with trivial canonical bundle has a vanishing Kobayashi pseudodistance.*

We refer to [**51**] for examples and discussions concerning Conjecture 0.3. A beautiful progress on this conjecture has been obtained recently by Verbitsky [**48**] in the case of hyper-Kähler manifolds (see also Diverio's contribution to this volume). The two conjectures together suggest a parallel between the Kodaira dimension of an algebraic variety and the degeneracy level of its Kobayashi pseudometric or pseudodistance. The great novelty of Campana's ideas developed in [**16**], [**17**] is to suggest that apart from the two extreme cases presented in the two conjectures above, the relationships between these two measures of positivity of the cotangent bundle is much more subtle. Indeed, Campana makes the following conjecture (cf. [**16**]);

***Conjecture 0.4***. – *Special varieties have vanishing Kobayashi pseudodistance.*

As we will discuss in Section 4, special varieties may have all possible Kodaira dimensions except for the maximal one, i.e., they cannot be of general type. For example, most elliptic fibrations over $\mathbb{P}^{n-1}$ will have Kodaira dimension $n-1$ and will be special.

An essential feature of projective or compact Kähler geometry which makes the second strategy work is the fact that parameter spaces for compact closed algebraic (or analytic) subsets of a projective (or compact Kähler) manifold is compact. We refer for Section 2.1 for details. If we drop the Kähler assumption, then the local deformation theory of closed analytic subschemes is unchanged, but we completely loose the compactness, as shows the example of the twistor family of a $K3$ surface $S$ (see Section 2.1, Example 2.2).

## 1. Fibrations and holomorphic forms

This section is devoted to the construction of rational fibrations using sections of line bundles or differential forms. The main applications concern the geometry of the canonical line bundle, with the beginning of a birational classification of algebraic varieties by the Kodaira dimension.

**1.1. General facts on fibrations and holomorphic forms. –** Let $f : X \to Y$ be a surjective morphism, where $X$ and $Y$ are smooth complex projective varieties or compact Kähler manifolds.

***Remark 1.1***. – Starting from a morphism $f : X \to Y$, up to replacing $f$ by its Stein factorization $f_{St} : X \to Y_{St}$, one may assume the fibers are connected, but then $Y_{St}$ is only normal and some extra work is needed if one also wants smoothness of $Y$.

The morphism $f$ is proper and we have:

***Lemma 1.2***. – *The following properties hold:*

(i) *The morphism $f_* : \pi_1(X) \to \pi_1(Y)$ has image of finite index (and is surjective if the fibers are connected).*

(ii) *The morphisms $f^* : H^i(Y, \mathbb{Q}) \to H^i(X, \mathbb{Q})$ are injective morphisms of Hodge structures.*

(iii) *The morphisms* $f^* : H^0(Y, \Omega_Y^i) \to H^0(X, \Omega_X^i)$ *are isomorphisms for* $i \geq 0$ *if the smooth fibers* $X_s$ *of* $f$ *are connected and satisfy* $H^0(X_s, \Omega_{X_s}^j) = 0$ *for all* $j$ *such that* $i \geq j > 0$.

*Proof.* – (i) follows from the fact that there is a dense Zariski (or analytic-Zariski) open set $U \subset Y$ such that the restriction $f_U : X_U \to U$ is smooth, hence a locally topologically trivial fibration. Then the statements hold for $f_{U*} : \pi_1(X_U) \to \pi_1(U)$ and we conclude using the fact the map $\pi_1(U) \to \pi_1(Y)$ is surjective because $Y$ is smooth.

For items (ii) and (iii), we use the fact that there is a left inverse on cohomology with real coefficients given by $\alpha \mapsto f_*(\omega^d \smile \alpha)$, where $d = \dim X - \dim Y$ is the relative dimension and $\omega$ is the class of a Kähler form on $X$ with volume 1 along the fibers of $f$. It follows that both maps $f^*$ are injective. Finally the proof of (iii) uses the cotangent bundle sequence along smooth fibers

$$0 \to f^*\Omega_{Y,s} \to \Omega_{X|X_s} \to \Omega_{X_s} \to 0, \tag{1}$$

which induces a filtration of $\Omega^i_{X|X_s}$, showing that under our assumptions

$$H^0(X_s, \Omega^i_{X|X_s}) = H^0(X_s, f^*\Omega^i_{Y_s}).$$

We then conclude that any holomorphic differential $i$-form on $X$ is a section of $f^*\Omega^i_Y$, at least over the open set $Y^0$ of regular values of $f$ and more precisely is of the form $f^*\beta$, where $\beta$ is a holomorphic differential $i$-form on $Y^0$. But then $\beta$ extends to $Y$ as $\beta' = f_*(\omega^d \wedge \alpha)$, and we thus conclude that $\alpha = f^*\beta'$ on $X^0 = f^{-1}(Y^0)$ hence everywhere. □

***Remark 1.3***. – The final argument given above fails if we replace holomorphic forms by pluridifferential holomorphic forms. We will see in Section 4 that for pluricanonical forms, it is not true that a section of $\Omega_X^{\otimes i}$, which is a section of $f^*\Omega_Y^{\otimes i} \subset \Omega_X^{\otimes i}$ along a dense Zariski open set $X_U = f^{-1}(U) \subset X$ is the pull-back of a section of $\Omega_Y^{\otimes i}$. However Lemma 1.2, (iii) is also true for pluridifferential forms.

***Remark 1.4***. – The exact sequence (1) is a particular case of the conormal bundle exact sequence

$$0 \to N^*_{X_s/X} \to \Omega_{X|X_s} \to \Omega_{X_s} \to 0, \tag{2}$$

where in this case $N_{X_s/X} = f^*T_{Y,s}$.

Note also the following standard fact which will apply more generally to a covering family of $X$ by $d$-dimensional varieties $\phi_s : \mathscr{X}_s \to X$, that can be seen as a diagram

$$\begin{array}{ccc} \mathscr{X} & \xrightarrow{\phi} & X \\ \downarrow{\scriptstyle f} & & \\ Y, & & \end{array} \tag{3}$$

where $f$ is a fibration and $\phi$ is dominant generically finite:

**Lemma 1.5.** – *In case of a fibration $f : X \to Y$, one has $K_{X|X_s} = K_{X_s}$. For a covering family $(\mathscr{X}_s)_{s\in Y}$, and for a general member $\mathscr{X}_s$ of the covering family, one has $K_{\mathscr{X}_s} = \phi_s^* K_X + D$, where $D$ is effective on $\mathscr{X}_s$.*

*Proof.* – The first statement follows from (1) by taking determinants, using the fact that $f^*\Omega_Y$ is trivial along the fiber $\mathscr{X}_s$. Only the second point needs to be proved. However we have by the first property $K_{\mathscr{X}_s} = K_{\mathscr{X}|\mathscr{X}_s}$ and on the other hand $K_{\mathscr{X}} = \phi^* K_X + R$, where $R \subset \mathscr{X}$ is the ramification divisor of $\phi$. Thus the statement holds once $\mathscr{X}_s$ is not contained in $R$. □

We end this section with a standard lemma that will be used (sometimes implicitly) throughout the paper:

**Lemma 1.6.** – *The holomorphic pluridifferential forms, that is sections of $\Omega_X^{\otimes k}$, $k \geq 0$, are bimeromorphic invariants of the compact complex manifold $X$.*

*Proof.* – Let $\phi : X \dashrightarrow Y$ be a bimeromorphic map, with $X$ and $Y$ compact. There exists a Zariski-analytic open set $U \subset X$ such that $\operatorname{codim} X \setminus U \geq 2$ and $\phi$ is well defined on $U$. We can thus define (because $k \geq 0$)

$$\phi^* : H^0(Y, \Omega_Y^{\otimes k}) \to H^0(U, \Omega_U^{\otimes k}).$$

This morphism is injective because $\phi$ is generically of maximal rank. By Hartogs' theorem, we have $H^0(U, \Omega_U^{\otimes k}) = H^0(X, \Omega_X^{\otimes k})$. We thus constructed an injective morphism $\phi^* : H^0(Y, \Omega_{Y/K}^{\otimes k}) \to H^0(X, \Omega_{X/K}^{\otimes k})$, which admits as inverse $(\phi^{-1})^*$. □

**1.2. Iitaka fibration.** – Let $X$ be a smooth projective variety and let $L$ be a line bundle on $X$. The subset $M(L) \subset \mathbb{N}$ defined as

$$M(L) = \{k \in \mathbb{N},\ H^0(X, kL) \neq 0\}$$

is a submonoid of $\mathbb{N}$ hence except for small values of $k$, it agrees with the set of positive multiples of an integer $k_0$. We assume from now on that $k_0 \neq 0$. The integer $k_0$ has the property that for any sufficiently large number $m$ divisible by $k_0$, there are nonzero sections of $mL$. Typical examples where $k_0$ is actually needed, that is, powers of $L$ of order nondivisible by $k_0$ have no nonzero sections, are given by $X = E \times Y$, where $E$ is an elliptic curve, and $L = L_0 \boxtimes L_Y$, where $L_0$ is a torsion line bundle of order $k_0$ on $E$ and $L_Y$ is an ample line bundle on $Y$. The Iitaka dimension $\kappa(X, L)$ of $(X, L)$ is defined as $-\infty$ if $M(L) = \{0\}$ and as the maximal dimension of the images of the maps

$$\phi_{kL} : X \dashrightarrow \mathbb{P}^N$$

induced by the linear systems $|kL|$ for $k \in M(L)$. For example $\kappa(X, L) = 0$ is equivalent to the fact that $M(L) \neq \{0\}$ and $h^0(X, kL) = 1$ for any $k \in M(L)$. When $L = K_X$, the Iitaka dimension of $L$ is the Kodaira dimension of $X$.