Six Lectures on Motives

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by

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Preface

These lecture notes are taken from my lecture series in the Asian-French summer school on motives and related topics. My goal in my lectures was two-fold: to give first of all a sketch of Voevodsky's foundational construction of the triangulated category of motives and its basic properties, and then to give an idea of some of the applications and wider vistas this construction has made possible. In doing this, I wanted also to point out some of the origins of this theory, coming from both the categorical side involving aspects of sheaf theory and triangulated categories, as well as the input from algebraic geometry, mainly through algebraic cycles. This latter aspect led me to devote an entire lecture to so-called moving lemmas, as I felt this subject captured much of the geometric side of the theory. I also reviewed much of the necessary material about triangulated categories and sheaves on a Grothendieck site, with the intention of making the discussion as accessible as possible.

I chose mixed Tate motives for illustrating applications. This topic touches on a broad range of subjects, including the theory of t-structures, Tannakian categories, rational homotopy theory, Grothendieck-Teichmüller theory, moduli of curves, polylogarithms and multiple zeta values. For this reason, I felt that an overview would be of interest to a fairly wide audience. I have also included two lectures on the extension of motives given by the motivic stable homotopy category of Morel-Voevodsky, giving a sketch of the construction as well as a discussion of the motivic Postnikov tower.

Many thanks are due to Joël Riou, whose careful reading and thoughtful comments allowed me to correct quite a few errors, as well as greatly improving the exposition.

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Other than this, I have made only minor changes and additions to my original lectures in these notes; I hope this will transmit the informal nature of the lectures to the reader. The rewriting of these notes lets me recall how much I enjoyed the summer school at the I.H.E.S and gives me the opportunity of thanking most heartily the organizers of summer school, Jean-Marc Fontaine and Jean-Benoit Bost, for putting together a truly worthwhile conference.

Marc Levine
Essen, December 2006 and October 2010

LECTURE I
TRIANGULATED CATEGORIES OF MOTIVES

Our goal in this lecture is to construct a *category of motives* that should capture the fundamental properties and structures of a reasonable cohomology theory on smooth varieties over a field k. To guide the construction, we ask the rather vague question: what kind of structures does "cohomology" have? At the very least, one should have

1. Pull-back maps $f^* : H^*(Y) \to H^*(X)$ maps $f : X \to Y$.
2. Products $H^*(X) \times H^*(Y) \to H^*(X \times Y)$.
3. Some long exact sequences: for example, Mayer-Vietoris for (Zariski) open covers.
4. Some isomorphisms, for example $H^*(X) \cong H^*(X \times \mathbb{A}^1)$.

Next, what categorical constructions will lead to all these structures? First of all, there is an algebraic machinery for generating long exact sequences and imposing isomorphisms, namely the machinery of triangulated categories. This structure is a result of axiomatizing the basic example of the derived category of an abelian category. For example, if one considers the abelian category Shv_T of sheaves of abelian groups on a topological space T, then the sheaf cohomology $H^*(T, A)$ with coefficients in an abelian group A is given as the Ext-group

$$H^n(T, A) \cong \mathrm{Ext}^n_{\mathrm{Shv}_T}(\mathbb{Z}_T, A_T),$$

where $\mathbb{Z}_T$, A_T are the constant sheaves with value $\mathbb{Z}, A$. In the derived category $D(\mathrm{Shv}_T)$, one has the canonical isomorphism

$$\mathrm{Ext}^n(\mathbb{Z}_T, A_T) \cong \mathrm{Hom}_{D(\mathrm{Shv}_T)}(\mathbb{Z}_T, A_T[n]).$$

All the well-known long exact sequences for cohomology, such as the Mayer-Vietoris sequence, or the Bockstein sequence, arise from the long exact sequence machinery encoded in the triangulated category $D(\mathrm{Shv}_T)$. In a general triangulated category D, one can define the cohomology of an object X with values in another object A by

$$H^n(X, A) := \mathrm{Hom}_D(X, A[n]);$$

we shall see how the triangulated structure in D gives rise to lots of long exact sequences. This formal view of cohomology has proven extremely valuable in many areas of mathematics.

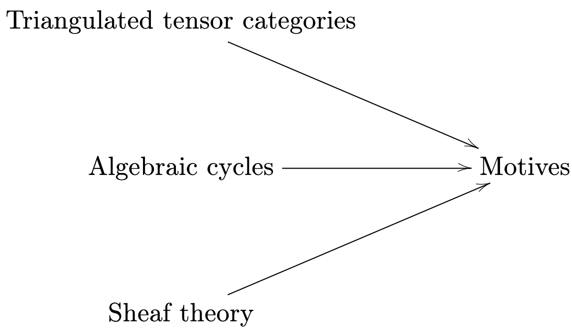
The product in cohomology comes from a tensor structure in the triangulated category D, namely a bi-functor

$$\otimes_D : D \times D \to D$$

with certain exactness properties. If our coefficient group A has a multiplication $A \otimes A \to A$ and our object X has a "diagonal" $\delta : X \to X \otimes X$, then our formal cohomology becomes a ring via

$$\begin{aligned}\mathrm{Hom}_D(X, A[n]) \otimes_{\mathbb{Z}} \mathrm{Hom}_D(X, A[m]) &\xrightarrow{\otimes_D} \mathrm{Hom}_D(X \otimes X, A[n+m]) \\ &\xrightarrow{\delta^*} \mathrm{Hom}_D(X, A[n+m]).\end{aligned}$$

We need some geometric input to feed this machine, coming from algebraic cycles. Finally, to understand what comes out of this construction, we need the homological algebra of sheaf theory. Schematically, we have the following picture:



1. Triangulated categories

1.1. Translations and triangles

Definition 1.1. – A *translation* on an additive category $\mathcal{A}$ is an equivalence $T : \mathcal{A} \to \mathcal{A}$. We write $X[1] := T(X)$. An additive functor $F : \mathcal{A} \to \mathcal{B}$ between additive categories with translation is *graded* if $F(X[1]) = (FX)[1]$ and similarly for morphisms.

Let $\mathcal{A}$ be an additive category with translation. A *triangle* (X, Y, Z, a, b, c) in $\mathcal{A}$ is a sequence of maps

$$X \xrightarrow{a} Y \xrightarrow{b} Z \xrightarrow{c} X[1].$$

A morphism of triangles

$$(f, g, h) : (X, Y, Z, a, b, c) \to (X', Y', Z', a', b', c')$$

is a commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{a} & Y & \xrightarrow{b} & Z & \xrightarrow{c} & X[1] \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle f[1]} \\ X' & \xrightarrow[a']{} & Y' & \xrightarrow[b']{} & Z' & \xrightarrow[c']{} & X'[1]. \end{array}$$