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REGULARIZED TRANSFORMATION-OPTICS CLOAKING IN ACOUSTIC AND ELECTROMAGNETIC SCATTERING

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Abstract. – We consider transformation-optics based cloaking in acoustic and electromagnetic scattering. The blueprints for an ideal cloak use singular acoustic and electromagnetic materials, posing severe difficulties to both theoretical analysis and practical fabrication. In order to avoid the singular structures, various regularized approximate cloaking schemes have been developed. We survey these developments in these lecture notes. We also propose some challenging issues for further investigation.

Résumé. – Nous étudions l'optique de transformation qui rend possible des méthodes de camouflage vis-à-vis des ondes acoustiques et électromagnétiques. Les dispositifs pour obtenir un camouflage idéal utilisent des matériaux acoustiques et électromagnétiques singuliers, ce qui pose de graves difficultés à la fois pour leur analyse et pour leur fabrication. Afin d'éviter de telles structures singulières, différentes méthodes de camouflage approximatif régularisées ont été développées. Nous présentons ces développements dans le présent document. Nous soulevons aussi quelques questions ouvertes difficiles pouvant faire l'objet de recherches ultérieures.

1. Introduction

We consider transformation-optics based cloaking in acoustic and electromagnetic scattering. The blueprints for an ideal cloak use singular acoustic and electromagnetic materials, posing server difficulties to both theoretical analysis and practical fabrication. In order to avoid the singular structures, various regularized approximate cloaking schemes have been developed. We survey these developments in this paper. We also propose some challenging issues for further investigation.

We shall be concerned with invisibility cloaking for acoustic and electromagnetic (EM) waves. A region is said to be *cloaked* if its content together with the cloak is indistinguishable from the background space with respect to exterior wave measurements. A proposal for cloaking for electrostatics using the invariance properties of

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the conductivity equation was pioneered in [**21, 22**]. Blueprints for making objects invisible to electromagnetic (EM) waves were proposed in two articles in *Science* in 2006 [**30, 46**]. The article by Pendry et al uses the same transformation used in [**21, 22**] while the work of Leonhardt uses a conformal mapping in two dimensions. The method based on the invariance properties of the equations modeling the wave phenomenon has been named *transformation optics* and has received a lot of attention in the scientific community and the popular press because of the generality of the method and its simplicity. There have been several other proposals for cloaking. We mention the works of Milton and Nicorovici [**42**] and of Alu and Engetha [**2**].

The method of transformation optics relies on the transformation properties of optical parameters and the transformation invariance of the governing wave equations. To obtain an ideal invisibility cloak, one first selects a region Ω in the background space for constructing the cloaking device. Throughout the paper, we assume that the background space is uniformly homogeneous in order to facilitate the exposition, but all of the results discussed in this paper can be straightforwardly extended to the case with an inhomogeneous background space. Let $P \in \Omega$ be a single point and let F be a diffeomorphism which blows up P to a region D within Ω. The ambient homogeneous medium around P is then 'compressed' via the *push-forward* of the transformation to form the cloaking medium in $\Omega\backslash\overline{D}$, whereas the 'hole' D is the cloaked region within which one can place the target object. The cloaking region $\Omega\backslash\overline{D}$ and the cloaked region D form the cloaking device in the *physical space*, whereas the homogeneous background space containing the singular point P is referred to as the *virtual space* (see Figure 3). Due to the transformation invariance of the corresponding wave equations, the acoustic/EM scattering in the physical space with respect to the cloaking device is the same as that in the virtual space. Heuristically speaking, the scattering information of the cloaking device is then 'hidden' in the singular point P. In a similar fashion cloaking devices based on blowing up a *crack* (namely, a curve in $\mathbb{R}^3$) or a *screen* (namely, a flat surface in $\mathbb{R}^3$) were, respectively, considered in [**16**] and [**34**], resulting in the so-called EM wormholes and carpet-cloak respectively.

The blow-up-a-point (respectively, -crack or -screen) construction yields singular cloaking materials, namely, the material parameters violate the regular conditions. The singular media present a great challenge for both theoretical analysis and practical fabrications. In order to tackle the acoustic and electromagnetic wave equations with singular coefficients underlying the ideal invisibility cloaks, finite energy solutions on Sobolev spaces with singular weights were introduced and studied in [**14, 16, 23, 40**]. On the other hand,several regularized constructions have been developed in the literature in order to avoid the singular structures. In [**15, 17, 47**], a truncation of singularities has been introduced. In [**26, 27, 35**], the blow-up-a-point transformation in [**22, 30, 46**] has been regularized to become the 'blow-up-a-small-region' transformation. Nevertheless, it is pointed out in [**25**] that the truncation-of-singularities construction and the blow-up-a-small-region construction are equivalent to each other. Instead of ideal/perfect invisibility, one would consider approximate/near invisibility for a regularized construction; that is, one intends to make the corresponding wave scattering effect due to a regularized cloaking device as small as possible depending on an asymptotically small regularization parameter $\rho \in \mathbb{R}_+$.

Due to its practical importance, the approximate cloaking has recently been extensively studied. In [**6, 27**], approximate cloaking schemes were developed for electrostatics. In [**5, 4, 26, 32, 33, 35, 36, 38, 44, 45**], various near-cloaking schemes were presented for scalar waves governed by the Helmholtz equation. In [**7, 8, 9, 39**], near-cloaking schemes were developed and investigated for the vector waves governed by the Maxwell system. Generally speaking, a regularized near-cloak consist of three layers: the innermost core is the cloaked region, the outermost layer is the cloaking region, and a compatible lossy layer right between the cloaked and cloaking regions. In the cloaking layer, the cloaking parameters are obtained by the push-forward construction mentioned earlier. Inside the cloaked region, from a practical viewpoint, one can place an arbitrary content, which could be both passive and active. The special lossy layer employed right between the cloaked and cloaking regions has shown to be necessary [**26, 39**], since otherwise there exist cloak-busting inclusions which defy any attempt for cloaking at particular resonant frequencies. In the extreme case when the lossy parameters go to infinity, the lossy layer become an impenetrable obstacle layer, and this is the one considered in [**5, 4, 7, 35**]. In the rest of this paper, we shall survey these developments and at certain places, we shall also point out challenges for further investigation. In addition to the present survey, we also refer to the survey papers [**10, 19, 18, 51, 52**] and the references therein for discussions of the theoretical and experimental progress on invisibility cloaking. In this paper we make emphasis on remote observations via the scattering amplitude or scattering operator. The same considerations are valid for the "near-field" which corresponds to the Cauchy data or the Dirichlet-to-Neumann map [**49**]. In Section 2, we review perfect cloaking for the case of electrostatics using Cauchy data. In Section 3 we define precisely what we mean by perfect cloaking for scattering. In Section 4 we review the push-forward construction. In Section 5 we consider regularized or approximate cloaks for acoustics, including the case of partial cloaks. In Section 6 we discuss regularized cloaks in electromagnetics.

2. Invisibility for electrostatics

We discuss here only perfect cloaking for electrostatics. For similar results for electromagnetic waves, acoustic waves, quantum waves, etc., see the review papers [**16**], [**19**] and the references given there. The fact that the boundary measurements do not change, when a conductivity is pushed forward by a smooth diffeomorphism leaving the boundary fixed can already be considered as a weak form of invisibility. Different media appear to be the same, and the apparent location of objects can change. However, this does not yet constitute real invisibility, as nothing has been hidden from view. In invisibility cloaking the aim is to hide an object inside a domain by surrounding it with a material so that even the presence of this object can not be detected by measurements on the domain's boundary. This means that all boundary measurements for the domain with this cloaked object included would be the same as if the domain were filled with a homogeneous, isotropic material. Theoretical models for this have been found by applying diffeomorphisms having singularities. These were first introduced in the framework of electrostatics, yielding counterexamples to the anisotropic Calderón problem (see [**50**] for a review) in the form of singular, anisotropic conductivities in $\mathbb{R}^n, n \geq 3$, indistinguishable from a constant isotropic conductivity in that they have the same Dirichlet-to-Neumann map [**21, 22**]. The same construction was rediscovered for electromagnetism in [**46**], with the intention of actually building such a device with appropriately designed metamaterials; a modified version of this was then experimentally demonstrated in [**48**]. (See also [**30**] for a somewhat different approach to cloaking in the high frequency limit.) The first constructions in this direction were based on blowing up the metric around a point [**28**]. In this construction, let (M, g) be a compact 2-dimensional manifold with non-empty boundary, let $x_0 \in M$ and consider the manifold

$$\widetilde{M} = M \setminus \{x_0\}$$

with the metric

$$\widetilde{g}_{ij}(x) = \frac{1}{d_M(x, x_0)^2} g_{ij}(x),$$

where $d_M(x, x_0)$ is the distance between x and x_0 on (M, g). Then $(\widetilde{M}, \widetilde{g})$ is a complete, non-compact 2-dimensional Riemannian manifold with the boundary $\partial\widetilde{M} = \partial M$. Essentially, the point x_0 has been "pulled to infinity". On the manifolds M and $\widetilde{M}$ we consider the boundary value problems

$$\begin{cases} \Delta_g u = 0 & \text{in } M, \\ u = f & \text{on } \partial M, \end{cases} \qquad \text{and} \qquad \begin{cases} \Delta_{\widetilde{g}} \widetilde{u} = 0 & \text{in } \widetilde{M}, \\ \widetilde{u} = f & \text{on } \partial\widetilde{M}, \\ \widetilde{u} \in L^\infty(\widetilde{M}). \end{cases}$$

Here Δ_g denotes the Laplace-Beltrami operator associated to the metric g. Note that in dimension $n \geq 3$ metrics and conductivities are equivalent. These boundary value problems are uniquely solvable and define the DN maps

$$\Lambda_{M,g} f = \partial_\nu u|_{\partial M}, \quad \Lambda_{\widetilde{M},\widetilde{g}} f = \partial_\nu \widetilde{u}|_{\partial\widetilde{M}}$$

where ∂_ν denotes the corresponding conormal derivatives. Since, in the two dimensional case, functions which are harmonic with respect to the metric g stay harmonic with respect to any metric which is conformal to g, one can see that $\Lambda_{M,g} = \Lambda_{\widetilde{M},\widetilde{g}}$. This can be seen using e.g., Brownian motion or capacity arguments. Thus, the boundary measurements for (M, g) and $(\widetilde{M}, \widetilde{g})$ coincide. This gives a counterexample for the inverse electrostatic problem on Riemannian surfaces – even the topology of possibly non-compact Riemannian surfaces can not be determined using boundary measurements (see Fig. 1). The above example can be thought as a "hole" in a Riemann surface that does not change the boundary measurements. Roughly speaking, mapping the manifold $\widetilde{M}$ smoothly to the set $M \setminus \overline{B}_M(x_0, \rho)$, where $B_M(x_0, \rho)$ is a metric ball of M,