Layer potential approaches to interface problems

Hyeonbae Kang

edited by

H. Ammari, J. Garnier

Panoramas et Synthèses

Numéro 44

SOCIÉTÉ MATHÉMATIQUE DE FRANCE
Publié avec le concours du Centre national de la recherche scientifique

Panoramas & Synthèses
44, 2014, p. 63–110

LAYER POTENTIAL APPROACHES TO INTERFACE PROBLEMS

by

Hyeonbae Kang

Abstract. – We review recent progress on imaging by generalized polarization tensors (GPTs), enhancement of near-cloaking by GPT-vanishing structures, cloaking by anomalous localized resonance, and analysis of stress concentration. These seemingly unrelated problems are all interface problems, and an integral operator called the Neumann-Poincaré operator naturally arises from them. We discuss about boundedness and invertibility properties, and spectral property of this operator, and then relate these properties with above mentioned problems.

Résumé. – Nous passons en revue les progrès récents de l'imagerie par tenseurs de polarisation généralisés (GPTs), la quasi-invisibilité pour des structures à GPTs évanescents, le camouflage (cloaking) par résonance localisée anormale, et l'analyse de la concentration de contraintes. Ces problèmes apparemment sans rapport sont tous des problèmes d'interface, et un opérateur intégral appelé opérateur de Neumann-Poincaré apparaît naturellement dans leur formulation. Nous discutons des bornes, des propriétés d'inversibilité, et des propriétés spectrales de cet opérateur, et nous établissons des liens entre ces propriétés et les problèmes mentionnés ci-dessus.

1. Introduction

This paper reviews recent progress on imaging by generalized polarization tensors (GPTs), enhancement of near-cloaking by GPT-vanishing structures, cloaking by anomalous localized resonance, and analysis of stress concentration. These seemingly unrelated problems are all interface problems, and an integral operator called the Neumann-Poincaré operator arises naturally from them. We discuss about boundedness and invertibility properties, and spectral property of this operator, and then relate these properties with above mentioned problems.

2010 Mathematics Subject Classification. – 35J47, 35R30, 35B30.
Key words and phrases. – Inverse problems, conductivity, cloaking.

2. Neumann-Poincaré operator

We begin our investigation by looking into the classical Neumann boundary value problem. Let Ω be a bounded domain in $\mathbb{R}^d$ (smoothness of the boundary $\partial\Omega$ will be specified later) and consider for a given Neumann data g the boundary value problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = g & \text{on } \partial\Omega. \end{cases} \tag{1}$$

Here and throughout this paper ν denotes the unit outward normal vector to $\partial\Omega$. We emphasize that g satisfies $\int_{\partial\Omega} g = 0$ for compatibility and the solution u is assumed to satisfy $\int_{\partial\Omega} u = 0$ to guarantee uniqueness of the solution.

A classical way of solving (1) is to use layer potentials. Let $\Gamma(x)$ be the fundamental solution to the Laplacian, i.e.,

$$\Gamma(x) = \begin{cases} \frac{1}{2\pi} \ln|x| \ , & d = 2 \ , \\ \frac{1}{(2-d)\omega_d} |x|^{2-d} \ , & d \geq 3 \ , \end{cases} \tag{2}$$

where ω_d denotes the area of the unit sphere in $\mathbb{R}^d$. The single layer potential $\mathcal{S}_{\partial\Omega}[\varphi]$ of a density function $\varphi \in L^2(\partial\Omega)$ is defined by

$$\mathcal{S}_{\partial\Omega}[\varphi](x) := \int_{\partial\Omega} \Gamma(x-y)\varphi(y)\, d\sigma(y) \ , \quad x \in \mathbb{R}^d. \tag{3}$$

If we set $u(x) = \mathcal{S}_{\partial\Omega}[\varphi](x)$ for some function φ, then u is harmonic in Ω. So in order for u to be the solution to (1), it suffices to choose φ so that the boundary condition is fulfilled.

The single layer potential $\mathcal{S}_{\partial\Omega}[\varphi]$ satisfies the jump relation

$$\frac{\partial}{\partial \nu} \mathcal{S}_{\partial\Omega}[\varphi]\big|_{\pm}(x) = \Big(\pm\frac{1}{2} I + \mathcal{K}^*_{\partial\Omega}\Big)[\varphi](x), \quad x \in \partial\Omega \ , \tag{4}$$

where the operator $\mathcal{K}_{\partial\Omega}$ is defined by

$$\mathcal{K}_{\partial\Omega}[\varphi](x) = \frac{1}{\omega_d} \int_{\partial\Omega} \frac{\langle y-x, \nu_y \rangle}{|x-y|^d} \varphi(y)\, d\sigma(y) \ , \quad x \in \partial\Omega, \tag{5}$$

and $\mathcal{K}^*_{\partial\Omega}$ is its L^2-adjoint, i.e.,

$$\mathcal{K}^*_{\partial\Omega}[\varphi](x) = \frac{1}{\omega_d} \int_{\partial\Omega} \frac{\langle x-y, \nu_x \rangle}{|x-y|^d} \varphi(y)\, d\sigma(y) \ . \tag{6}$$

Here $\pm$ indicates the limits (to $\partial\Omega$) from outside and inside of Ω, respectively. So in order to fulfill the boundary condition in (1), we need to solve the integral equation

$$\Big(-\frac{1}{2} I + \mathcal{K}^*_{\partial\Omega}\Big)[\varphi] = g \quad \text{on } \Omega. \tag{7}$$

The operator $\mathcal{K}_{\partial\Omega}$ (or $\mathcal{K}^*_{\partial\Omega}$) is called the Neumann-Poincaré (NP) operator associated with the domain Ω. It is well known (see for example **[21, 53, 78]**) that if $\partial\Omega$ is smooth ($\mathcal{C}^{1,\alpha}$ for some $\alpha > 0$), then

(i) $\mathscr{K}_{\partial\Omega}^*$ is a compact operator on $L^2(\partial\Omega)$,
(ii) the spectrum of $\mathscr{K}_{\partial\Omega}^*$ lies in $(-\frac{1}{2}, \frac{1}{2}]$,
(iii) $\frac{1}{2}I + \mathscr{K}_{\partial\Omega}^*$ is invertible on $L^2(\partial\Omega)$ and $-\frac{1}{2}I + \mathscr{K}_{\partial\Omega}^*$ is invertible on $L_0^2(\partial\Omega)$.

Here, $L_0^2(\partial\Omega)$ is the collection of all L^2 functions with the mean zero. If $g \in L_0^2(\partial\Omega)$ (to satisfy the compatibility condition), the solution to (7) is given by

$$\varphi = \Big(-\frac{1}{2}I + \mathscr{K}_{\partial\Omega}^*\Big)^{-1}[g],$$

and the solution to (1) by

$$u(x) = \mathscr{S}_{\partial\Omega}\Big(-\frac{1}{2}I + \mathscr{K}_{\partial\Omega}^*\Big)^{-1}[g](x), \quad x \in \Omega.$$

A few remarks on the above-mentioned properties of $\mathscr{K}_{\partial\Omega}^*$ are in order. If $\partial\Omega$ is $\mathscr{C}^{1,\alpha}$, then because of orthogonality of the normal vector and the tangential vector, we have

$$\frac{|\langle x-y, \nu_x\rangle|}{|x-y|^d} \le \frac{C}{|x-y|^{d-1-\alpha}}, \quad x, y \in \partial\Omega, \tag{8}$$

which makes $\mathscr{K}_{\partial\Omega}^*$ compact. Property (iii) can be proved using the Fredholm alternative. We emphasize that property (ii) holds not only for smooth domains but also for domains with Lipschitz boundaries. Even if we restrict our investigation here mostly to the domains with smooth boundaries, it is worth while to review two important results on the properties of the NP operators associated with Lipschitz domains. If $\partial\Omega$ is Lipschitz, then $\mathscr{K}_{\partial\Omega}^*$ is a singular integral operator, and L^2-boundedness of $\mathscr{K}_{\partial\Omega}^*$ was proved by Calderón [**44**] when the Lipschitz constant of $\partial\Omega$ is small, and by Coifman, McIntosh, and Meyer [**50**] for the general case. In this regards, it is worth mentioning $T[1]$ Theorem of David and Journé [**52**] which states that a singular integral operator T is bounded on L^2 if and only if $T[1]$ is a function of bounded mean oscillation. Invertibility as stated in (iii) for Lipschitz domains was proved by Verchota [**111**].

To motivate our discussion on the spectrum of the NP operator, we consider another problem: a transmission problem. Suppose that an inclusion Ω is immersed in the free space $\mathbb{R}^d$. Suppose that the conductivity (or the dielectric constant) of Ω is ϵ_c and that of the background is ϵ_m ($\epsilon_c \neq \epsilon_m$). So, the distribution of the conductivity is given by

$$\sigma_\Omega = \epsilon_c \chi(\Omega) + \epsilon_m \chi(\mathbb{R}^d \setminus \overline{\Omega}),$$

where χ denotes the indicator function. The problem we consider is

$$\begin{cases} \nabla \cdot \sigma_\Omega \nabla u = 0 & \text{in } \mathbb{R}^d, \\ u(x) - h(x) = O(|x|^{1-d}) & \text{as } |x| \to \infty, \end{cases} \tag{9}$$

for a given harmonic function h in $\mathbb{R}^d$. Note that without the inclusion Ω the solution to (9) is nothing but $u(x) = h(x)$. In the presence of the inclusion, the solution takes the form $u = h +$ something, and this something is generated by the discontinuity of the conductivity along $\partial\Omega$. It turns out (see for example **[21, 76]**) that there is a potential φ on $\partial\Omega$ such that the solution is given by

$$u(x) = h(x) + \mathcal{S}_{\partial\Omega}[\varphi](x), \quad x \in \mathbb{R}^d. \tag{10}$$

Since u satisfies the transmission conditions, $u|_- = u|_+$ (continuity of potential) and $\epsilon_c \frac{\partial u}{\partial \nu}|_- = \epsilon_m \frac{\partial u}{\partial \nu}|_+$ (continuity of flux), one can see from the jump relation (4) that the following relation holds:

$$\left(\frac{\epsilon_c + \epsilon_m}{2(\epsilon_c - \epsilon_m)} I - \mathcal{K}^*_{\partial\Omega} \right) [\varphi] = \frac{\partial h}{\partial \nu} \quad \text{on } \partial\Omega. \tag{11}$$

We emphasize that the problem (9) is elliptic if (and only if) ϵ_c and ϵ_m are positive, and in this case the number $\frac{\epsilon_c+\epsilon_m}{2(\epsilon_c-\epsilon_m)}$ does not belong to $[-1/2, 1/2]$, where the spectrum of $\mathcal{K}^*_{\partial\Omega}$ lies. So, as long as we are interested in elliptic problems, there is no need to look into the spectrum of $\mathcal{K}^*_{\partial\Omega}$. The spectrum of the NP operator is a classical subject of research since Poincaré. See a recent paper **[107]** and references therein for a brief history of this. Recently there has been renewed interest in the spectrum of the NP operator in relation to the plasmonic structures consisting of inclusions with negative dielectric constants, i.e., with $\epsilon_c < 0$ (while ϵ_m stays positive). In this case, $\frac{\epsilon_c+\epsilon_m}{2(\epsilon_c-\epsilon_m)}$ may lie in the spectrum of $\mathcal{K}^*_{\partial\Omega}$. As we will see in the next section, if $\partial\Omega$ is $\mathcal{C}^{1,\alpha}$ (so that $\mathcal{K}^*_{\partial\Omega}$ is compact), then the spectrum of $\mathcal{K}^*_{\partial\Omega}$ is discrete and accumulating to 0. The number $\frac{\epsilon_c}{\epsilon_m}$ such that $\frac{\epsilon_c+\epsilon_m}{2(\epsilon_c-\epsilon_m)}$ is an eigenvalue of $\mathcal{K}^*_{\partial\Omega}$ is called a plasmonic eigenvalue and the single layer potential of the corresponding eigenfunction is called a localized plasmon **[59]**.

2.1. Spectrum of the NP operator. – We first emphasize that $\mathcal{K}^*_{\partial\Omega}$ is not self-adjoint on the usual L^2-space. In fact, it is self-adjoint on $L^2(\partial\Omega)$ only if Ω is a disk or a ball **[91]**. However, we may realize $\mathcal{K}^*_{\partial\Omega}$ as a self-adjoint operator by using a different inner product.

Let $\langle \, , \, \rangle$ be the usual inner product on $L^2(\partial\Omega)$. It is easy to see that $\mathcal{S}_{\partial\Omega}$ is self-adjoint on $L^2(\partial\Omega)$, which is nothing but saying $\Gamma(x-y) = \Gamma(y-x)$. Let $\varphi \in L^2_0(\partial\Omega)$ and define

$$u(x) = \mathcal{S}_{\partial\Omega}[\varphi](x), \quad x \in \mathbb{R}^d. \tag{12}$$

Then $u(x) = O(|x|^{1-d})$ as $|x| \to \infty$, and we have

$$\int_\Omega |\nabla u|^2 dx = \int_{\partial\Omega} u \Big(-\frac{1}{2}\varphi + \mathcal{K}^*_{\partial\Omega}[\varphi] \Big) d\sigma,$$
$$\int_{\mathbb{R}^d \setminus \overline{\Omega}} |\nabla u|^2 dx = -\int_{\partial\Omega} u \Big(\frac{1}{2}\varphi + \mathcal{K}^*_{\partial\Omega}[\varphi] \Big) d\sigma.$$

Summing up these two identities we find

$$\int_{\mathbb{R}^d} |\nabla u|^2 dx = -\langle \varphi, \mathcal{S}_{\partial\Omega}[\varphi] \rangle. \tag{13}$$