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THE DIMER MODEL IN STATISTICAL MECHANICS

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Abstract. – The dimer model is the study of random perfect matchings of a planar graph, representing the adsorption of diatomic molecules on the surface of a crystal. It belongs to the field of statistical mechanics. The goal for these lectures is to give an overview of: the foundational results of Kasteleyn, Temperley and Fisher, proving an explicit formula for the partition funtion; Thurston's height function interpretation of dimer configurations of bipartite graphs; the paper "Dimers and Amoebae" of Kenyon, Okounkov and Sheffield, giving a full description of the dimer model on infinite, bi-periodic, bipartite graphs.

Résumé. – Le modèle de dimères consiste en l'étude de couplages parfaits aléatoires d'un graphe planaire, représentant la répartition de molécules di-atomiques à la surface d'un cristal. Ce modèle appartient au domaine plus large de la mécanique statistique. Le but de ces notes de cours est de donner un aperçu des résultats suivants : expression explicite pour la fonction de partition due à Kasteleyn, Temperley et Fisher ; interprétation en tant que fonction de hauteur des configurations de dimères d'un graphe biparti due à Thurston ; description complète du modèle de dimères sur un graphe infini, bi-périodique et biparti, due à Kenyon, Okounkov et Sheffield.

1. Introduction

1.1. Statistical mechanics and 2-dimensional models. – *Statistical mechanics* is the application of *probability theory*, which includes mathematical tools for dealing with large populations, to the field of *mechanics*, which is concerned with the motion of

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particles or objects when subjected to a force. Statistical mechanics provides a framework for relating the microscopic properties of individual atoms and molecules to the macroscopic bulk properties of materials that can be observed in everyday life (source: 'Wikipedia').

In other words, statistical mechanics aims at studying large scale properties of physics system, based on probabilistic models describing microscopic interactions between components of the system. Statistical mechanics is also known as *statistical physics*.

It is a priori natural to introduce 3-dimensional graphs in order to accurately model the molecular structure of a material as for example a piece of iron, a porous material or water. Since the 3-dimensional version of many models turns out to be hardly tractable, much effort has been put into the study of their 2-dimensional counterpart. The latter have been shown to exhibit rich, complex and fascinating behaviors. Here are a few examples.

- *Percolation.* This model describes the flow of a liquid through a porous material. The system considered is a square grid representing the molecular structure of the material. Each bond of the grid is either "open" with probability p, or "closed" with probability $1-p$, and bonds are assumed to behave independently from each other. The set of open bonds in a given configuration represents the part of the material wetted by the liquid, and the main issue addressed is the existence and properties of infinite clusters of open edges. The behavior of the system depends on the parameter p: when $p = 0$, all edges are closed, there is no infinite cluster and the liquid cannot flow through the material; when $p = 1$, all edges are open and there is a unique infinite cluster filling the whole grid. One can show that there is a specific value of the parameter p, known as *critical* p, equal to $1/2$ for the square grid, below which the probability of having an infinite cluster of open edges is 0, and above which the probability of it existing and being unique is 1. One says that the system undergoes a *phase transition* at $p = 1/2$. References **[33, 18, 3, 53, 52]** are books or lecture notes giving an overview of percolation theory.
- The *Ising model.* The system considered is a magnet made of particles restricted to stay on a grid. Each particle has a spin which points either "up" or "down" (spin ± 1). Each configuration σ of spins on the whole grid has an energy $\mathcal{E}(\sigma)$, which is the sum of an interaction energy between pairs of neighboring spins, and of an interaction energy of spins with an external magnetic field. The probability of a configuration σ is proportional to $e^{-\frac{1}{kT}\mathcal{E}(\sigma)}$, where k is the Boltzmann constant, and T is the external temperature. When there is no magnetic field and the temperature is close to 0, spins tend to align with their neighbors and a typical configuration consists of all $+1$ or all -1. When the temperature is very high, all configurations have the same probability of occurring and a typical configuration consists of a mixture of $+1$ and -1. Again, there is a *critical temperature* T_c at which the Ising model undergoes a phase transition between

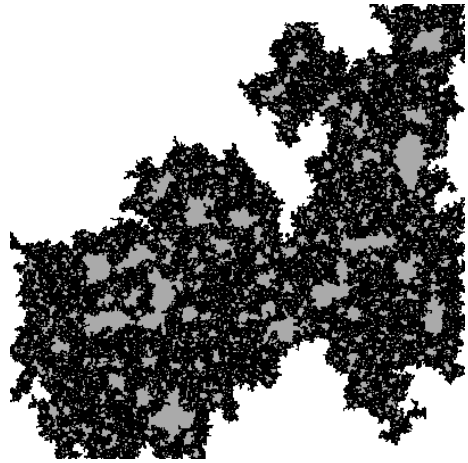

FIGURE 1. An infinite cluster of open edges, when $p = \frac{1}{2}$. Courtesy of V. Beffara.

the ordered and disordered phase. The literature on the Ising model is huge, as an introductory reading we would suggest the book by Baxter [**1**], the one by McCoy and Wu [**37**], the lecture notes by Velenik [**51**] and references therein.

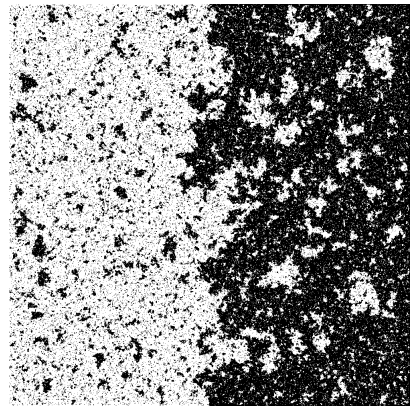

FIGURE 2. An Ising configuration, when $\frac{1}{T} = 0.9$. Courtesy of V. Beffara.

These two examples illustrate some of the principal challenges of 2-dimensional statistical mechanics, which are:

- Find the critical parameters of the models.
- Understand the behavior of the model in the sub-critical and super-critical regimes.

– Understand the behavior of the system at criticality. Critical systems exhibit surprising features, and are believed to be universal in the scaling limit, *i.e.,* independent of the specific features of the lattice on which the model is defined. Very precise predictions were established by physicists in the last 30-50 years, in particular by Nienhuis, Cardy, Duplantier and many others. On the mathematics side, a huge step forward was the introduction of the Schramm-Loewner evolution by Schramm in [**42**], a process conjectured to describe the limiting behavior of well chosen observables of critical models. Many of these conjectures were solved in the following years, in particular by Lawler, Schramm, Werner [**35**] and Smirnov [**45**], Chelkak-Smirnov [**6**]. The importance of these results was acknowledged with the two Fields medals awarded to Werner (2006) and Smirnov (2010). Interesting collaborations between the physics and mathematics communities are emerging, with for example the work of Duplantier and Sheffield [**12**].

The general framework for statistical mechanics is the following. Consider an object G (most often a graph) representing the physical system, and define all possible configurations of the system. To every configuration σ, assign an energy $\mathscr{E}(\sigma)$, then the probability of occurrence of the configuration σ is given by the *Boltzmann* measure μ:

$$\mu(\sigma) = \frac{e^{-\mathscr{E}(\sigma)}}{Z(\mathsf{G})}.$$

Note that the energy is often multiplied by a parameter representing the inverse external temperature. The denominator $Z(\mathsf{G})$ is the normalizing constant, known as the *partition function*:

$$Z(\mathsf{G}) = \sum_{\sigma} e^{-\mathscr{E}(\sigma)}.$$

When the system is infinite, the above definition does not hold, but we do not want to enter into these considerations here.

The partition function is one of the key objects of statistical mechanics. Indeed it encodes much of the macroscopic behavior of the system. Hence, its computation is the first question one addresses when studying such a model. It turns out that there are very few models where this computation can be done exactly. Having a closed form for the partition function opens the way to finding many exact results, and to having a very deep understanding of the macroscopic behavior of the system.

Two famous examples are the 2-dimensional Ising model, where the computation of the partition function is due to Onsager [**40**], and the *dimer model* where it is due to Kasteleyn [**20, 21**], and independently to Temperley and Fisher [**47**]. The dimer model is the main topic of these lectures and is defined in the next section.

1.2. The dimer model. – The dimer model was introduced in the physics and chemists communities to represent the adsorption of di-atomic molecules on the surface of a crystal. It is part of a larger family of models describing the adsorption of molecules of different sizes on a lattice. It was first mentioned in a paper by Fowler and Rushbrooke