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François GOLSE & Maria Pia GUALDANI & Cyril IMBERT & Alexis VASSEUR

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Annales Scientifiques de l'École Normale Supérieure,
45, rue d'Ulm, 75230 Paris Cedex 05, France.
Tél. : (33) 1 44 32 20 88. Fax : (33) 1 44 32 20 80.
Email : `annales@ens.fr`

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Société Mathématique de France
Case 916 - Luminy
13288 Marseille Cedex 09
Tél. : (33) 04 91 26 74 64. Fax : (33) 04 91 41 17 51
Email : `abonnements@smf.emath.fr`

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PARTIAL REGULARITY IN TIME FOR THE SPACE-HOMOGENEOUS LANDAU EQUATION WITH COULOMB POTENTIAL

BY François GOLSE, Maria Pia GUALDANI, Cyril IMBERT AND Alexis VASSEUR

ABSTRACT. – We prove that the set of singular times for weak solutions of the space-homogeneous Landau equation with Coulomb potential constructed as in [C. Villani, *Arch. Rational Mech. Anal.* **143** (1998), 273–307] has Hausdorff dimension at most $\frac{1}{2}$.

RÉSUMÉ. – Nous démontrons que l'ensemble des temps singuliers pour les solutions faibles de l'équation de Landau homogène en espace avec potentiel colombien construites comme dans [C. Villani, *Arch. Rational Mech. Anal.* **143** (1998), 273–307] est de dimension de Hausdorff au plus $\frac{1}{2}$.

1. Introduction

We are concerned with the regularity of weak solutions $f \equiv f(t,v) \geq 0$ a.e. to the space-homogeneous Landau equation with Coulomb interaction potential

$$\partial_t f(t,v) = \operatorname{div}_v \int_{\mathbf{R}^3} a(v-w)(f(t,w)\nabla_v f(t,v) - f(t,v)\nabla_w f(t,w))dw, \quad v \in \mathbf{R}^3, \tag{1}$$

where the collision kernel a is the matrix field

$$a(z) = \frac{\nabla^2|z|}{8\pi} = \frac{\Pi(z)}{8\pi|z|}, \quad \text{with } \Pi(z) := I - \left(\frac{z}{|z|}\right)^{\otimes 2}.$$

(In other words, $\Pi(z)$ is the orthogonal projection on $(\mathbf{R}z)^\perp$ for all $z \in \mathbf{R}^3 \setminus \{0\}$.) This equation is used in the description of collisions between charged particles in plasma physics (see [24] or §41 in [26]).

Equivalently, the Landau equation with Coulomb potential takes the form

$$\partial_t f(t,v) = \operatorname{trace}(A[f](t,v)\nabla_v^2 f(t,v)) + f(t,v)^2 \tag{2}$$

with $A[f](t,\cdot) := a \star f(t,\cdot)$.

Villani has proved the global existence of a special kind of weak solutions of the Cauchy problem for (1), known as "H-solutions," for all initial data with finite mass, energy and entropy (Theorem 4 (i) in [35]). Whether H-solutions of (1) with smooth initial data remain

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smooth for all times or blow up in finite time is one of the outstanding problems in the mathematical analysis of kinetic models: see §1.3 (2) in Chapter 5 of Villani's monograph [36]. The form (2) of the Landau equation suggests that blow-up might occur in finite time, by analogy with the semilinear heat equation $\partial_t u(t,x) = \Delta_x u(t,x) + u(t,x)^2$: see the last statement in Theorem 1 of [37] (for nonnegative solutions of the initial boundary value problem on a smooth bounded domain of $\mathbf{R}^3$ with homogeneous Dirichlet condition at the boundary), or Section 5.4 in [7]. Nevertheless, global existence of classical, radially symmetric and nonincresing (in the velocity variable) solutions has been established for the equation $\partial_t u(t,x) = ((-\Delta_x)^{-1}u)(t,x)\Delta_x u(t,x) + \alpha u(t,x)^2$ which can be seen as an "isotropic" variant of (2) (i.e., with the diffusion matrix $A[f]$ replaced with the diffusion coefficient $(-\Delta_v)^{-1}f$ and the constant 1 replaced with the (smaller) coefficient α), for all $\alpha \in [0, \frac{74}{75})$ in [15, 23], then for $\alpha = 1$ in [17].

These arguments being somewhat inconclusive, the recent research on the Cauchy problem for (1) has produced mostly conditional results — with one notable exception, which we shall discuss in more detail below. For instance, the uniqueness of bounded solutions of (1) has been proved in [12]; the regularity of radial L^p solutions with $p > \frac{3}{2}$ has been proved in [17] (see also [16]); the case of nonradial L^p solutions with $p > \frac{3}{2}$ and moments in v of order larger than 8 is treated in [30], while the large time behavior of H-solutions of (1) is discussed in [5]. Of course, the existence and uniqueness theory for the space-inhomogeneous Landau equation is even harder, and most of the existing results on that equation bear on near-Maxwellian equilibrium global solutions [18, 6, 8], apart from the very general weak stability result in [28]. There are also various local existence and uniqueness results, as well as smoothing estimates for solutions of the space-inhomogeneous Landau equation under the assumption of locally (in time and space) bounded moments in v: see [19, 21, 20] (notice that [21, 20] require only that the distribution function has bounded moments in v of order larger than $9/2$). Since the present paper is focused on the Coulomb case, which is the most interesting on physical grounds, we have omitted the rather large literature on the generalizations of the Landau equation where the collision kernel a is replaced with $|z|^{\gamma+2}\Pi(z)$ with $\gamma > -3$.

Perhaps the most remarkable recent contribution to the mathematical theory of (1) is the following result by Desvillettes [10]: any (nonnegative) H-solution of (1) on $[0,T] \times \mathbf{R}^3$ with finite mass, energy and entropy satisfies the bound

$$\int_0^T \int_{\mathbf{R}^3} \frac{|\nabla_v \sqrt{f(t,v)}|^2}{(1+|v|)^3} dv dt < \infty\,; \tag{3}$$

(see Theorem 1 in [10]).This bound implies in particular the propagation of moments in v of arbitrary order for such solutions of (1) (see Proposition 2 in [10]). Both this bound and the propagation of moments are of key importance in the present work.

Our main result is the following partial regularity statement, which will be presented and discussed in detail in the next section.

Main Theorem. – *The set of singular times of any H-solution of* (1) *constructed by the approximation scheme described in* [35] *has Hausdorff dimension at most* $\frac{1}{2}$.

2. Main Results

The prototype of all partial regularity results in partial differential equations is Leray's observation [25] that the set of singular times of any Leray solution to the Navier-Stokes equations for incompressible fluid dynamics in three space dimensions has Hausdorff dimension at most $\frac{1}{2}$ (see §34 in [25], especially, formula (6.5)). Leray's remark was later considerably refined by Scheffer [29], and by Caffarelli, Kohn and Nirenberg [4] (see also [32, 27, 34])

One key ingredient in Leray's observation is the energy inequality satisfied by Leray solutions to the Navier-Stokes equations. Our first task is therefore to establish an analogous inequality for solutions to (1). Henceforth we denote by $L^p_k(\mathbf{R}^3)$ the set of measurable $g \equiv g(v)$ defined a.e. on $\mathbf{R}^3$ such that

$$\|g\|_{L^p_k(\mathbf{R}^3)} := \left(\int_{\mathbf{R}^3} (1+|v|^2)^{k/2}|g(v)|^p dv\right)^{1/p} < \infty.$$

We first recall that an H-solution to (1) on the time interval $[0,T)$ with initial data $f_{\text{in}} \equiv f_{\text{in}}(v) \geq 0$ a.e. is an element $f \in C([0,T);\mathcal{D}'(\mathbf{R}^3)) \cap L^1((0,T);L^1_{-1}(\mathbf{R}^3))$ such that $\Pi(v-w)(\nabla_v - \nabla_w)\sqrt{f(t,v)f(t,w)/|v-w|} \in L^2([0,T]\times\mathbf{R}^3_v\times\mathbf{R}^3_w)$ for each $T>0$, satisfying

$$\text{(4)}\quad f(t,v) \geq 0 \text{ for a.e. } v \in \mathbf{R}^3, \quad \text{and} \quad \int_{\mathbf{R}^3}\begin{pmatrix}1\\ v\\ |v|^2\end{pmatrix} f(t,v)dv = \int_{\mathbf{R}^3}\begin{pmatrix}1\\ v\\ |v|^2\end{pmatrix} f_{\text{in}}(t,v)dv,$$

while

$$\text{(5)}\qquad \int_{\mathbf{R}^3} f(t,v)\ln f(t,v)dv \leq \int_{\mathbf{R}^3} f_{\text{in}}(v)\ln f_{\text{in}}(v)dv$$

for a.e. $t \geq 0$, and

$$\begin{aligned}&\int_{\mathbf{R}^3} f_{\text{in}}(v)\phi(0,v)dv + \int_0^T\int_{\mathbf{R}^3} f(t,v)\partial_t\phi(t,v)dv\\ &= \int_0^T\int_{\mathbf{R}^3}\sqrt{\tfrac{f(t,v)f(t,w)}{8\pi|v-w|}}(\nabla\phi(t,v)-\nabla\phi(t,w))\cdot\Pi(v-w)(\nabla_v-\nabla_w)\sqrt{\tfrac{f(t,v)f(t,w)}{8\pi|v-w|}}dvdw\end{aligned}$$

for each $\phi \in C^2_c([0,T)\times\mathbf{R}^3)$. Observe that the integral on the right-hand side of the previous equality exists since it is assumed that

$$\Pi(v-w)(\nabla_v-\nabla_w)\sqrt{f(t,v)f(t,w)/|v-w|} \in L^2([0,T]\times\mathbf{R}^3_v\times\mathbf{R}^3_w),$$

while

$$\sqrt{f(t,v)f(t,w)/|v-w|}(\nabla\phi(v)-\nabla\phi(w)) \in L^2([0,T]\times\mathbf{R}^3_v\times\mathbf{R}^3_w)$$

(see equation (42) in [35]). Of course, the notion of H-solution is based on the observation that classical solutions of the Landau equation with appropriate decay as $|v| \to +\infty$ satisfy

$$\text{(6)}\quad \frac{d}{dt}H(f)(t) = -\tfrac{1}{2}\iint_{\mathbf{R}^3\times\mathbf{R}^3}\frac{f(t,v)f(t,w)}{|v-w|}\Pi(v-w) : (\nabla_v \ln f(t,v) - \nabla_w \ln f(t,w))^{\otimes 2}dvdw \leq 0,$$

where

$$H(f)(t) := \int_{\mathbf{R}^3} f(t,v)\ln f(t,v)dv.$$