COURS SPÉCIALISÉS 25

# LECTURES ON ELLIPTIC METHODS FOR HYBRID INVERSE PROBLEMS

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# PREFACE

The starting point for these lectures is a course given in Paris between January and March 2014 as part of Chaire Junior of the Fondation Sciences Mathématiques de Paris. This book is designed for a graduate audience, interested in inverse problems and partial differential equations, and we have tried to make it as self-contained as possible.

The analysis of hybrid imaging problems relies on several areas of the theory of PDE together with tools often used to study inverse problems. The full description of the models involved, from the theoretical foundations to the most current developments, would require several volumes and is beyond the scope of these notes, which we designed of a size commensurate with a twenty hour lecture course, the original format of the course. The presentation is limited to simplified settings, so that complete results could be explained entirely. This allows us to provide a proper course, instead of a survey of current research, but it comes at the price that more advanced results are not presented. We have tried to give references to some of the major seminal papers in the area in the hope that the interested reader would then follow these trails to the most current advances by means of usual bibliographical reference libraries.

The physical model most often encountered in this book is the linear Maxwell system of equations. It is of foremost importance in the physics of inverse electromagnetic problems. Compared to the conductivity equation and the Helmholtz equation, the analysis of Maxwell's system is much less developed, and these lectures contain several new results which have been established while writing this book. In the chapter discussing regularity properties, we focus on the Maxwell system of equations in the time harmonic case. Proofs regarding small volume inhomogeneities are given for Maxwell's system as well.

We introduce the inverse source problem from time-dependent boundary measurements for the wave equation from the classical control theory point of view, leaving aside many deep results related to the geometric control of the wave equation or the Radon transform, or recent developments concerning randomised data. Probabilistic methods are not used, random media are not considered, compressed sensing and other image processing approaches are not mentioned. All these questions would certainly be perfectly natural in this course, but would require a different set of authors. For many of these questions, we refer the reader to the relevant chapters of the Handbook of Mathematical Methods in Imaging [**192**] for detailed introductions and references.

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# CHAPTER 1

# INTRODUCTION

The inverse problems we discuss are the non-physical counterparts of physics based direct problems. A direct problem is a model of the link from cause to effect, and in this course we shall focus on direct problems modelled by partial differential equations where the effects of a cause are uniquely observable, that is, well posed problems in the sense of Hadamard: from an initial or boundary condition, there exists a unique solution, which depends continuously on the input data [**109**].

Inverse problems correspond to the opposite problem, namely to find the cause which generated the observed, measured result. Such problems are almost necessarily ill-posed (and therefore non physical). As absolute precision in a measure is impossible, measured data are always (local) averages. A field is measured on a finite number of sensors, and is therefore only known partially. One could say that making a measure which is faithful, that is, which when performed several times will provide the same result, implies filtering small variations, thus applying a compact operator to the full field. Reconstructing the cause from measurements thus corresponds to the inversion of a compact operator, which is necessarily unbounded and thus unstable, except in finite dimension. Schematically, starting from $A$, a cause (the parameters of a PDE, a source term, an initial condition), which is transformed into $B$, the solution, by the partial differential equation, and then into $C$, the measured trace of the solution, the inversion from $C$ to $B$ is always unstable, whereas the inversion of $B$ to $A$ may be stable or unstable depending on the nature of the PDE, but $B \to A$ is often less severely ill posed than $C \to B$.

As a first fundamental example, let us consider the electrical impedance tomography (EIT) problem, also known as the Calderón problem in the mathematics literature.

## 1.1. The electrical impedance tomography problem

**1.1.1. Measurements on the exterior boundary: the Calderón problem. —** Let $\Omega \subseteq \mathbb{R}^d$ be a Lipschitz connected bounded domain, where $d \geq 2$ is the dimension of the ambient space.

We consider a real-valued conductivity coefficient $\sigma \in L^\infty(\Omega)$, satisfying

$$\Lambda^{-1} \leq \sigma(x) \leq \Lambda \quad \text{for almost every } x \in \Omega \tag{1.1}$$

for some constant $\Lambda > 0$.

***Definition 1.1.*** — The *Dirichlet to Neumann map* is

$$\Lambda_\sigma : H^{1/2}(\partial\Omega) \longrightarrow H^{-1/2}(\partial\Omega), \quad \langle \Lambda_\sigma \varphi, \psi \rangle = \int_\Omega \sigma \nabla u \cdot \nabla v \, \mathrm{d}x,$$

where $v \in H^1(\Omega)$ is such that $v_{|\partial\Omega} = \psi$ and $u \in H^1(\Omega)$ is the weak solution of

$$\begin{cases} -\operatorname{div}(\sigma \nabla u) = 0 & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega. \end{cases}$$

We need to "prove" this definition, because it apparently depends on the choice of the test function $v$. Given $v_1, v_2 \in H^1(\Omega)$ with the same trace, namely $v_1 - v_2 \in H_0^1(\Omega)$, from the definition of weak solution we have

$$\int_\Omega \sigma \nabla u \cdot \nabla(v_1 - v_2) \, \mathrm{d}x = 0,$$

thus this definition is proper.