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Nouvelle série **Ext-ALGEBRA OF $SL_2(\mathbb{Q}_p)$**

R. OLLIVIER & P. SCHNEIDER

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**ON THE PRO- p IWAHORI HECKE
EXT-ALGEBRA OF $SL_2(\mathbb{Q}_p)$**

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ON THE PRO- p IWAHORI HECKE EXT-ALGEBRA OF $\mathrm{SL}_2(\mathbb{Q}_p)$

Rachel Ollivier, Peter Schneider

Abstract. – Let $G = \mathrm{SL}_2(\mathfrak{F})$ where \mathfrak{F} is a finite extension of \mathbb{Q}_p . We suppose that the pro- p Iwahori subgroup I of G is a Poincaré group of dimension d . Let k be a field containing the residue field of \mathfrak{F} .

In this volume, we study the graded Ext-algebra $E^* = \mathrm{Ext}_{\mathrm{Mod}(G)}^*(k[G/I], k[G/I])$. Its degree zero piece E^0 is the usual pro- p Iwahori-Hecke k -algebra H .

We study E^d as an H -bimodule and deduce that for an irreducible admissible smooth k -representation V of G , we have $H^d(I, V) = 0$ unless V is the trivial representation.

When $\mathfrak{F} = \mathbb{Q}_p$ with $p \geq 5$, we have $d = 3$. In that case we describe E^* as an H -bimodule and give the structure as an algebra of the centralizer in E^* of the center of H . We deduce results on the values of the functor $H^*(I, -)$ which attaches to a (finite length) smooth k -representation V of G its cohomology with respect to I . We prove that $H^*(I, V)$ is always finite dimensional. Furthermore, if V is irreducible, then V is supersingular if and only if $H^*(I, V)$ is a supersingular H -module.

Résumé (Sur la Ext-algèbre de Hecke du pro- p Iwahori de $\mathrm{SL}_2(\mathbb{Q}_p)$)

Soit $G = \mathrm{SL}_2(\mathfrak{F})$ où \mathfrak{F} est une extension finie \mathbb{Q}_p . On suppose que le sous-groupe d'Iwahori I de G est un groupe de Poincaré de dimension d . Soit k un corps contenant le corps résiduel de \mathfrak{F} .

Dans ce texte, nous étudions la Ext-algèbre graduée $E^* = \mathrm{Ext}_{\mathrm{Mod}(G)}^*(k[G/I], k[G/I])$. Sa composante de degré zero est la k -algèbre de Hecke du pro- p Iwahori H .

Nous étudions le H -bimodule E^d et déduisons que, étant donnée une k -représentation irréductible admissible lisse V de G , on a $H^d(I, V) = 0$ à moins que V ne soit la représentation triviale.

Lorsque $\mathfrak{F} = \mathbb{Q}_p$ avec $p \geq 5$, on a $d = 3$. Dans ce cas, nous décrivons le H -bimodule E^* et la structure d'algèbre du centralisateur dans E^* du centre de H . Nous en déduisons des résultats quant aux valeurs du foncteur qui attache à une k -représentation lisse (de longueur finie) V de G l'espace de I -cohomologie $H^*(I, V)$. Nous montrons que $H^*(I, V)$ est toujours de dimension finie. De plus, si V est irréductible, alors V est supersingulière si et seulement si $H^*(I, V)$ est un module supersingulier.

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CHAPTER 1

INTRODUCTION

Let \mathfrak{F} be a locally compact nonarchimedean field with residue characteristic p , and let G be the group of \mathfrak{F} -rational points of a connected reductive group \mathbf{G} over \mathfrak{F} . We suppose that \mathbf{G} is \mathfrak{F} -split.

Let k be a field of characteristic p and let $\text{Mod}(G)$ denote the category of all smooth representations of G in k -vector spaces. For a general \mathbf{G} and \mathfrak{F} this category is still poorly understood. One way of approaching it consists in considering the Hecke algebra H of the pro- p Iwahori subgroup $I \subset G$. In this case the natural left exact functor

$$\begin{aligned} \mathfrak{h} : \text{Mod}(G) &\longrightarrow \text{Mod}(H) \\ V &\longmapsto V^I = \text{Hom}_{k[G]}(\mathbf{X}, V) \end{aligned}$$

sends a nonzero representation onto a nonzero module. Its left adjoint is

$$\begin{aligned} \mathfrak{t} : \text{Mod}(H) &\longrightarrow \text{Mod}^I(G) \subseteq \text{Mod}(G) \\ M &\longmapsto \mathbf{X} \otimes_H M. \end{aligned}$$

Here \mathbf{X} denotes the space of k -valued functions with compact support on G/I with the natural left action of G . The functor \mathfrak{t} has values in the category $\text{Mod}^I(G)$ of all smooth k -representations of G generated by their I -fixed vectors. This category, which a priori has no reason to be an abelian subcategory of $\text{Mod}(G)$, contains all irreducible representations. But in general \mathfrak{t} is not an equivalence of categories and little is known about $\text{Mod}^I(G)$ and $\text{Mod}(G)$ unless $G = \text{GL}_2(\mathbb{Q}_p)$ or $G = \text{SL}_2(\mathbb{Q}_p)$ ([6], [11], [13], [16]).

The functor \mathfrak{h} , although left exact, is not right exact since p divides the pro-order of I . It is therefore natural to consider the derived functor. In [17] the following result is shown: When \mathfrak{F} is a finite extension of \mathbb{Q}_p and I is a torsion free pro- p group, there exists a derived version of the functors \mathfrak{h} and \mathfrak{t} providing an equivalence between the derived category $D(G)$ of smooth representations of G in k -vector spaces and the derived category of differential graded modules over a certain differential graded pro- p Iwahori-Hecke algebra H^\bullet .

The article [14] opened up the study of the Hecke differential graded algebra H^\bullet by giving the first results on its cohomology algebra $E^* := \text{Ext}_{\text{Mod}(G)}^*(\mathbf{X}, \mathbf{X})$. This is the pro- p Iwahori Hecke Ext-algebra we refer to in the title of the current article. We suppose in this introduction that I is a torsion free p -adic Lie group which forces \mathfrak{F} to be a finite extension of \mathbb{Q}_p . We denote by d the dimension of I as a Poincaré group. The Ext algebra E^* is supported in degrees 0 to d .

When \mathbf{G} is almost simple and simply connected, the ideal $\mathfrak{J}H$ which controls the supersingularity (see §2.1) has finite codimension in H . We show that we have an isomorphism of H -bimodules

$$(1) \quad \text{Ext}_{\text{Mod}(G)}^d(\mathbf{X}, \mathbf{X}) \cong \chi_{\text{triv}} \oplus \text{Inj}((H/\mathfrak{J}H)^\vee),$$

where χ_{triv} is the trivial character of H and $\text{Inj}((H/\mathfrak{J}H)^\vee)$ is an injective envelope of the dual module $(H/\mathfrak{J}H)^\vee$. When $\mathbf{G} = \text{SL}_2$, the center of H contains a polynomial algebra $k[\zeta]$ and $\mathfrak{J}H = \zeta H$. The large injective module inside of $\text{Ext}_{\text{Mod}(G)}^d(\mathbf{X}, \mathbf{X})$ is ξ -divisible for any $\xi \in H$ which is a non-zero-divisor. This, together with the decomposition (1), allows us to prove (Cor. 2.19) that given Q a nonzero polynomial in $k[X]$, we have $H^d(I, \mathbf{X}/\mathbf{X}Q(\zeta)) = 0$ unless $Q(1) = 0$ in which case $H^d(I, \mathbf{X}/\mathbf{X}Q(\zeta)) \cong \chi_{\text{triv}}$. But we remark that every irreducible admissible representation of $\text{SL}_2(\mathfrak{F})$ is a quotient $\mathbf{X}/\mathbf{X}Q(\zeta)$ for some Q as above and we prove:

PROPOSITION (Proposition 2.20). – *We have $H^d(I, V) = 0$ for any irreducible admissible representation of $\text{SL}_2(\mathfrak{F})$ except when V is the trivial representation in which case $H^d(I, k_{\text{triv}}) \cong \chi_{\text{triv}}$ as an H -bimodule.*

In Sections 3 and 4, we move on to the study of E^1 and E^{d-1} respectively. Here we fully use the Frobenius reciprocity recalled in §2.2 which allows to identify E^i with $H^i(I, \mathbf{X})$. We decompose the latter, via the Shapiro isomorphism, as a direct sum

$$\bigoplus_{w \in \widetilde{W}} H^i(I_w, k)$$

where w ranges over \widetilde{W} (defined at the beginning of Section 2, see also §2.4.1) and $I_w = I \cap wIw^{-1}$. We explain in §3.2 that we see elements of $H^1(I_w, k)$ as triples. This is valid for $G = \text{SL}_2(\mathfrak{F})$ with no restriction on \mathfrak{F} and stems from the computation of the Frattini quotient of I_w . When I is a Poincaré group of dimension d , we use the duality between E^1 and E^{d-1} (§14) to also express elements of $H^{d-1}(I_w, k)$ as triples in §4.1. When $G = \text{SL}_2(\mathbb{Q}_p)$, $p \geq 5$, Remark §3.2 points out that the triples of $H^1(I_w, k)$ are simply the elements in

$$\text{Hom}(\mathbb{Z}_p/p\mathbb{Z}_p, k) \times \text{Hom}((1+p\mathbb{Z}_p)/(1+p^2\mathbb{Z}_p), k) \times \text{Hom}(\mathbb{Z}_p/p\mathbb{Z}_p, k)$$

hence by duality the triples of $H^2(I_w, k)$ are the elements in

$$\mathbb{Z}_p/p\mathbb{Z}_p \otimes_{\mathbb{F}_p} k \times ((1+p\mathbb{Z}_p)/(1+p^2\mathbb{Z}_p)) \otimes_{\mathbb{F}_p} k \times \mathbb{Z}_p/p\mathbb{Z}_p \otimes_{\mathbb{F}_p} k.$$

In this context, the full left action of H on the triples of E^1 and of E^2 can be found in §3.6 and §4.3 (the proof of the most technical formulas is postponed to the appendix).

The right action of H on the triples can be deduced using the anti-involution \mathcal{J} of E^* (see §2.2.3 and Lemmas 3.7 and 4.1). We are especially interested in the left and right action of the central element $\zeta \in H$ (which is fixed by \mathcal{J}).

In Section 5 we study the $k[\zeta]$ -torsion on the left in certain graded pieces of E^* when $G = \mathrm{SL}_2(\mathfrak{F})$, with various restrictive conditions on \mathfrak{F} depending on the graded piece in question. Only for the computation of the $k[\zeta]$ -torsion in E^2 do we use the explicit formulas for the action of ζ hence we have to restrict ourselves to $G = \mathrm{SL}_2(\mathbb{Q}_p)$, $p \geq 5$.

Contemplating the formulas for the action of ζ on E^1 and E^2 (still when $G = \mathrm{SL}_2(\mathbb{Q}_p)$, $p \geq 5$) emphasizes the role of the operators

$$f := \zeta \cdot \mathrm{id}_{E^*} \cdot \zeta - \mathrm{id}_{E^*} \quad \text{and} \quad g := \zeta \cdot \mathrm{id}_{E^*} - \mathrm{id}_{E^*} \cdot \zeta$$

as introduced in §6.1. The kernel of f is a $k[\zeta^{\pm 1}]$ -bimodule. Describing its structure as an H -bimodule requires the technical Paragraph 3.7.3.2 (then see Propositions 6.8, 6.19 and Lemma 6.2). On the other hand, as the centralizer in E^* of ζ , the kernel of g is naturally a subalgebra of E^* . We describe this kernel precisely in §6.2.1 and §6.3.1 (and Lemma 6.2) and conclude in Proposition 8.1 that it actually coincides with the centralizer $\mathcal{C}_{E^*}(Z)$ of the whole center $Z := Z(H)$ of H in E^* . The product in this natural subalgebra of E^* is explicitly given in Section 8. (Note that the center of H is no longer central in E^*).

Proposition 6.13 says that E^2 is, as an H -bimodule, isomorphic to the direct sum of the kernels of the operators f and g (restricted to E^2) and Proposition 6.10 says that it is also (essentially) the case for E^1 . This allows us to completely determine the structure of E^* as a left and right $k[\zeta]$ -module (Proposition 7.2) and to establish results such as Proposition 7.6 where we study the $k[\zeta]$ -torsion on the left in spaces of the form $H^*(I, \mathbf{X}/\mathbf{X}Q(\zeta))$ for $Q \in k[X]$. This in particular leads to the following theorem:

THEOREM (Theorem 7.11). – *Let $G = \mathrm{SL}_2(\mathbb{Q}_p)$ with $p \neq 2, 3$. For any representation of finite length in $\mathrm{Mod}(G)$ we have:*

- i. *The k -vector space $H^*(I, V)$ is finite dimensional;*
- ii. *if V is generated by its subspace of I -fixed vectors V^I and $Q(\zeta)V^I = 0$ for some nonzero polynomial $Q \in k[X]$, then the left H -module $H^*(I, V)$ is $P(\zeta)$ -torsion for the polynomial $P(X) := Q(X)Q(\frac{1}{X})X^{\deg(Q)}$.*

The most interesting consequence of this theorem is that, under the same hypotheses, an irreducible representation V in $\mathrm{Mod}(G)$ is supersingular if and only if the left H -module $H^*(I, V)$ is supersingular (this is Corollary 7.12 which uses the theorem in the case when $Q = X$). This strongly indicates that the notion of supersingularity for general G can be extended to objects in the derived category $D(G)$ by introducing a theory of supports via the dg algebra H^\bullet . We hope to return to this in another paper.

In [13] §3.5 we studied the representation theoretic meaning of the localization H_ζ of the Hecke algebra in the central element ζ . Despite the fact that ζ is no longer central in E^* it turns out (Remark 7.7) that $\zeta^{\mathbb{N}_0}$ is a left and right Ore set in E^* , so

that the localization E_ζ^* does exist. We will show elsewhere that E_ζ^* again is a Yoneda Ext-algebra and will investigate its meaning for the nonsupersingular $\mathrm{SL}_2(\mathbb{Q}_p)$ -representations.

After this paper was finished E. Bodon ([2]) gave in his thesis, building very much on the computational methods developed in the present paper, two further structural results in the case $G = \mathrm{SL}_2(\mathbb{Q}_p)$ with $p \neq 2, 3$. He describes explicitly the full graded center of E^* . Even more remarkably he shows that the algebra E^* as an algebra over H is finitely presented.

In forthcoming work of the second author with K. Ardakov we develop a general theory of central spaces for a certain class of Grothendieck categories which refines the notion of the center of an abelian category. It was shown in [1] that the usual center of the category $\mathrm{Mod}(G)$ is very small. For example, if $\mathbf{G} = \mathrm{SL}_2$ then this center is the group ring $k[Z(G)]$ of the center of G . In contrast the central space in this case with $\mathfrak{X} = \mathbb{Q}_p$ is a projective variety over k which is a quotient of the affine variety $\mathrm{Spec}(Z)$ by a relation which is given by the annihilator ideal of the $Z \otimes_k Z$ -bimodule E^* . The results of the present paper allow to compute this ideal and therefore this projective variety explicitly. Therefore we strongly believe that this bimodule and its support variety play a basic role for the computation of the central space of $\mathrm{Mod}(G)$ for general groups G .

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