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THE BREUIL-MÉZARD CONJECTURE FOR NON-SCALAR SPLIT RESIDUAL REPRESENTATIONS

BY YONGQUAN HU AND FUCHENG TAN

ABSTRACT. – We prove the Breuil-Mézard conjecture for split non-scalar residual representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ by local methods. Combined with the cases previously proved in [20] and [26], this completes the proof of the conjecture (when $p \geq 5$). As a consequence, the local restriction in the proof of the Fontaine-Mazur conjecture in [20] is removed.

RÉSUMÉ. – Nous prouvons la conjecture de Breuil-Mézard pour les représentations résiduelles scindées non-scalaires de $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ par des méthodes locales. Combiné avec les cas déjà prouvés dans [20] et [26], cela complète la preuve de la conjecture (lorsque $p \geq 5$). Par conséquent, la restriction locale dans la preuve de la conjecture de Fontaine-Mazur dans [20] est levée.

Notation

- $p \geq 5$ is a prime number. The p-adic valuation is normalized as $v_p(p) = 1$.
- $E/\mathbb{Q}_p$ is a sufficiently large finite extension with ring of integers $\mathcal{O}$, a (fixed) uniformizer ϖ, and residue field $\mathbb{F}$. Its subring of Witt vectors is denoted by $W(\mathbb{F})$.
- For a number field F, the completion at a place v is written as F_v, for which we fix a uniformizer denoted by ϖ_v.
- For a local or global field L, $G_L = \mathrm{Gal}(\overline{L}/L)$. The inertia subgroup for the local field is written as I_L.
- For each finite place v in a number field F, fix a map $G_{F_v} \to G_F$ by choosing an inclusion $\overline{F} \hookrightarrow \overline{F}_v$ of algebraic closures.
- $\epsilon : G_{\mathbb{Q}_p} \to \mathbb{Z}_p^\times$ is the cyclotomic character, $\omega : G_{\mathbb{Q}_p} \to \mathbb{F}_p^\times$ is its reduction mod p, and $\tilde{\omega}$ is the Teichmüller lifting of ω.
- $\mathbb{1} : G_{\mathbb{Q}_p} \to \mathbb{F}_p^\times$ is the trivial character. We also let $\mathbb{1}$ denote other trivial representations, if no confusion arises.
- Normalize the local class field map $\mathbb{Q}_p^\times \to G_{\mathbb{Q}_p}^{\mathrm{ab}}$ so that uniformizers correspond to geometric Frobenii. Then a character of $G_{\mathbb{Q}_p}$ will also be regarded as a character of $\mathbb{Q}_p^\times$.
- For a ring R, m-Spec$R$ denotes the set of maximal ideals.

- For R a Noetherian ring and M a finite R-module of dimension at most d, let $\ell_{R_{\mathfrak{p}}}(M_{\mathfrak{p}})$ denote the length of the $R_{\mathfrak{p}}$-module $M_{\mathfrak{p}}$, and let $\mathcal{Z}_d(M) = \sum_{\mathfrak{p}} \ell_{R_{\mathfrak{p}}}(M_{\mathfrak{p}})\mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Spec} R$ such that $\dim R/\mathfrak{p} = d$. When the context is clear, we simply denote it by $\mathcal{Z}(M)$.
- For R a Noetherian local ring with maximal ideal $\mathfrak{m}$ and M a finite R-module, and for an $\mathfrak{m}$-primary ideal $\mathfrak{q}$ of R, let $e_{\mathfrak{q}}(R, M)$ denote the Hilbert-Samuel multiplicity of M with respect to $\mathfrak{q}$. We abbreviate $e_{\mathfrak{m}}(R, M) = e(R, M)$ and $e_{\mathfrak{q}}(R, R) = e_{\mathfrak{q}}(R)$.
- For $r \geq 0$, we let $\mathrm{Sym}^r E^2$ (resp. $\mathrm{Sym}^r \mathbb{F}^2$) be the usual symmetric power representation of $\mathrm{GL}_2(\mathbb{Z}_p)$ (resp. of $\mathrm{GL}_2(\mathbb{F}_p)$, but viewed as a representation of $\mathrm{GL}_2(\mathbb{Z}_p)$).

1. Introduction

Consider the following data:

- an integer $k \geq 2$,
- a representation $\tau : I_{\mathbb{Q}_p} \to \mathrm{GL}_2(E)$ with open kernel,
- a continuous character $\psi : G_{\mathbb{Q}_p} \to \mathcal{O}^\times$ such that $\psi|_{I_{\mathbb{Q}_p}} = \epsilon^{k-2}\det\tau$.

We call such a triple (k, τ, ψ) a p-adic Hodge type. We say a 2-dimensional continuous representation $\rho : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(E)$ is of type (k, τ, ψ) if ρ is potentially semi-stable (i.e., de Rham) such that its Hodge-Tate weights are $(0, k-1)$, $\mathrm{WD}(\rho)|_{I_{\mathbb{Q}_p}} \simeq \tau$, and $\det\rho \simeq \psi\epsilon$. Here $\mathrm{WD}(\rho)$ is the Weil-Deligne representation associated to ρ by Fontaine [12].

By a result of Henniart [14], there is a unique finite dimensional smooth irreducible $\overline{\mathbb{Q}}_p$-representation $\sigma(\tau)$ (resp. $\sigma^{\mathrm{cr}}(\tau)$) of $\mathrm{GL}_2(\mathbb{Z}_p)$ associated to τ, such that for any infinite dimensional smooth absolutely irreducible representation π of $\mathrm{GL}_2(\mathbb{Q}_p)$ and the associated Weil-Deligne representation $LL(\pi)$ via classical local Langlands correspondence, we have $\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Z}_p)}(\sigma(\tau), \pi) \neq 0$ if and only if $LL(\pi)|_{I_{\mathbb{Q}_p}} \simeq \tau$ (resp. $\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Z}_p)}(\sigma^{\mathrm{cr}}(\tau), \pi) \neq 0$ if and only if $LL(\pi)|_{I_{\mathbb{Q}_p}} \simeq \tau$ and the monodromy operator is trivial). We remark that $\sigma(\tau)$ and $\sigma^{\mathrm{cr}}(\tau)$ differ only when $\tau = \chi \oplus \chi$ is scalar, in which case

$$\sigma(\tau) = \tilde{\mathrm{st}} \otimes \chi \circ \det, \quad \sigma^{\mathrm{cr}}(\tau) = \chi \circ \det$$

where $\tilde{\mathrm{st}}$ is the inflation to $\mathrm{GL}_2(\mathbb{Z}_p)$ of the Steinberg representation of $\mathrm{GL}_2(\mathbb{F}_p)$.

Enlarging E if needed, we may and do assume $\sigma(\tau)$ is defined over E. Form the finite dimensional $\mathrm{GL}_2(\mathbb{Z}_p)$-representation

$$\sigma(k, \tau) = \mathrm{Sym}^{k-2} E^2 \otimes_E \sigma(\tau)$$

and the semi-simplification $\overline{\sigma(k,\tau)}^{\mathrm{ss}}$ of the reduction modulo ϖ of a $\mathrm{GL}_2(\mathbb{Z}_p)$-stable $\mathcal{O}$-lattice inside $\sigma(k, \tau)$. Then $\overline{\sigma(k,\tau)}^{\mathrm{ss}}$ does not depend on the choice of the lattice.

Recall that the finite dimensional irreducible $\mathbb{F}$-representations of $\mathrm{GL}_2(\mathbb{Z}_p)$ are of the form

$$\sigma_{n,m} := \mathrm{Sym}^n \mathbb{F}^2 \otimes \det{}^m, \quad n \in \{0, \ldots, p-1\}, m \in \{0, \ldots, p-2\}.$$

For each $\sigma_{n,m}$ let $a_{n,m} = a_{n,m}(k, \tau)$ be the multiplicity with which $\sigma_{n,m}$ occurs in $\overline{\sigma(k,\tau)}^{\mathrm{ss}}$. We have the obvious analogue in the crystalline case by considering

$$\sigma^{\mathrm{cr}}(k, \tau) := \mathrm{Sym}^{k-2} E^2 \otimes_E \sigma^{\mathrm{cr}}(\tau)$$

and denote the resulting numbers by $a^{\mathrm{cr}}_{n,m} = a^{\mathrm{cr}}_{n,m}(k, \tau)$.

Let $\overline{\rho} : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(\mathbb{F})$ be a continuous representation and $R^{\square}(\overline{\rho})$ be its universal framed deformation ring ([19]). The following results on the structure of potentially semi-stable framed deformation rings are known.

THEOREM 1.1 (Kisin, [19]). – *There is a unique (possibly trivial) quotient* $R^{\square,\psi}(k,\tau,\overline{\rho})$ *(resp.* $R^{\square,\psi}_{\mathrm{cr}}(k,\tau,\overline{\rho})$*) of* $R^{\square}(\overline{\rho})$ *such that*

(i) *A map* $x : R^{\square}(\overline{\rho}) \to E'$*, for any finite extension* E'/E*, factors through* $R^{\square,\psi}(k,\tau,\overline{\rho})$ *(resp.* $R^{\square,\psi}_{\mathrm{cr}}(k,\tau,\overline{\rho})$*) if and only if the Galois representation* ρ_x *corresponding to* x *is of type* (k,τ,ψ) *(resp. and is potentially crystalline).*

(ii) $R^{\square,\psi}(k,\tau,\overline{\rho})$ *(resp.* $R^{\square,\psi}_{\mathrm{cr}}(k,\tau,\overline{\rho})$*) is* p*-torsion free.*

(iii) $R^{\square,\psi}(k,\tau,\overline{\rho})[1/p]$ *(resp.* $R^{\square,\psi}_{\mathrm{cr}}(k,\tau,\overline{\rho})[1/p]$*) is reduced, all of whose irreducible components are smooth of dimension* 4.

The following conjecture, the so-called Breuil-Mézard conjecture, relates the Hilbert-Samuel multiplicity of $R^{\square,\psi}(k,\tau,\overline{\rho})/\varpi$ (resp. $R^{\square,\psi}_{\mathrm{cr}}(k,\tau,\overline{\rho})/\varpi$) with the numbers $a_{n,m}$ (resp. $a^{\mathrm{cr}}_{n,m}$).

CONJECTURE 1.2 (Breuil-Mézard, [4]). – *For any* (k,τ,ψ) *as above, we have*

$$e(R^{\square,\psi}(k,\tau,\overline{\rho})/\varpi) = \sum_{n,m} a_{n,m}(k,\tau)\mu_{n,m}(\overline{\rho}), \tag{1}$$

$$e(R^{\square,\psi}_{\mathrm{cr}}(k,\tau,\overline{\rho})/\varpi) = \sum_{n,m} a^{\mathrm{cr}}_{n,m}(k,\tau)\mu_{n,m}(\overline{\rho}) \tag{2}$$

for some integers $\mu_{n,m}(\overline{\rho})$ *which are independent of* k*,* τ *and* ψ*.*

In particular, the conjecture implies that

$$\mu_{n,m}(\overline{\rho}) = e\left(R^{\square,\psi}_{\mathrm{cr}}(n+2,(\tilde{\omega}^m)^{\oplus 2},\overline{\rho})/\varpi\right)$$

which can be computed. We refer the reader to [20, 1.1.6] for these numbers, and remark that when $n = p-2$ and $\overline{\rho}$ is scalar, $\mu_{p-2,m}(\overline{\rho}) = 4$, as is shown in [28].

Conjecture 1.2 was proved by Kisin [20] in the cases that $\overline{\rho}$ is not (a twist of) an extension of $\mathbb{1}$ by ω. He first proved the "$\leq$" part of (1) and (2) using the p-adic local Langlands [6], and then combined it with the (global) modularity lifting method to deduce the "$\geq$" part. Years later, the conjecture was proved by Paškūnas [26] for all $\overline{\rho}$ with only scalar endomorphisms, using the p-adic local Langlands and his previous (local) results in [25]. We prove, also using local methods (except for one global input due to Emerton [9], see the introduction of [26]), the following theorem (in the language of cycles of [10]), which in particular includes the remaining case of the conjecture (when $p \geq 5$).

THEOREM 1.3 (Remark 5.7, Theorem 5.11, Theorem 5.12). – *For any continuous representation* $\overline{\rho} : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(\mathbb{F})$ *which is isomorphic to the direct sum of two distinct characters, and for any* (k,τ,ψ) *as above, there are* 4*-dimensional cycles* $\mathcal{Z}_{n,m}$ *of* $R^{\square}(\overline{\rho})$ *which are independent of* (k,τ,ψ) *such that*

$$\mathcal{Z}(R^{\square,\psi}(k,\tau,\overline{\rho})/\varpi) = \sum_{n,m} a_{n,m}(k,\tau)\mathcal{Z}_{n,m}.$$