DERIVED ALGEBRAIC GEOMETRY

D. Ben-Zvi, D. Calaque, J. Grivaux, E. Mann,
D. Nadler, T. Pantev, M. Robalo,
P. Safronov, G. Vezzosi

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christian.smf@cirm-math.fr

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75231 Paris Cedex 05, France
Tél : (33) 01 44 27 67 99 • Fax : (33) 01 40 46 90 96
panoramas@smf.emath.fr • http://smf.emath.fr/

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Société mathématique de France

David Ben-Zvi
Department of Mathematics University of Texas 2515 Speedway Stop C1200 Austin TX 78712, USA

David Nadler
Department of Mathematics, University of California, Berkeley, CA 94720-3840, USA

Étienne Mann
Université dAngers, Département de mathématiques, Bâtiment I, Faculté des Sciences, 2 Boulevard Lavoisier, F-49045 Angers Cedex 01, France

Marco Robalo
Sorbonne Université. Université Pierre et Marie Curie, Institut Mathématiques de Jussieu Paris Rive Gauche, CNRS, Case 247, 4, place Jussieu, F-75252 Paris Cedex 05, France

Gabriele Vezzosi
Dipartimento di Matematica ed Informatica U. Dini, Viale Morgagni, 67/a, 50134 Firenze, Italy

Tony Pantev
Department of Mathematics, University of Pennsylvania, DRL 209 South 33rd Street, Philadelphia, PA 19104-6395, USA

Damien Calaque
IMAG, Univ Montpellier, CNRS, Montpellier, France & Institut Universitaire de France

Julien Grivaux
IMJ-PRG, UMR 7586, Sorbonne Université, Case 247, 4 place Jussieu, F-75252 Paris Cedex 05, France

Pavel Safronov
School of Mathematics, University of Edinburgh, James Clerk Maxwell Building, Peter Guthrie Tait Road, Edinburgh UK, EH9 3FD

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***Abstract.* —** The present volume covers the content of one of the session of "États de la recherche" on derived algebraic geometry which has been held in Toulouse in June 2017. It contains the contributions of David BenZvi and David Nadler, Damien Calaque and Julien Grivaux, Etienne Mann and Marco Robalo, Tony Pantev and Gabriele Vezzosi, and Pavel Safronov, taken from their original lectures. These cover a wide variety of subjects, from foundations of derived algebraic geometry and derived deformation theory, to its applications to enumerative geometry, geometric representation theory and categorification in algebraic geometry.

***Résumé* (Géométrie algébrique dérivée). —** Le présent volume est un compte rendu d'une des sessions des « États de la recherche » sur la géométrie algébrique dérivée qui s'est tenue à Toulouse en juin 2017. Il contient les contributions de David BenZvi et David Nadler, Damien Calaque et Julien Grivaux, Etienne Mann et Marco Robalo, Tony Pantev et Gabriele Vezzosi, et de Pavel Safronov, associées à leurs mini-cours respectifs. Elles couvrent une grande variété de sujets du domaine, depuis les fondements de la géométrie algébrique dérivée et de la théorie des déformation, jusqu'à leurs applications à la géométrie énumérative, la théorie géométrique des représentations et à la catégorification dans le contexte de la géométrie algébrique.

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RÉSUMÉS DES ARTICLES

Introduction à la géométrie algébrique dérivée
Tony Pantev & Gabriele Vezzosi 1

Nous donnons une introduction à la géométrie algébrique dérivée (DAG) en se concentrant sur les constructions et techniques de base. Nous discutons les schémas dérivés affines, les champs algébriques dérivés et le théorème de représentabilité d'Artin-Lurie. À travers l'exemple des déformations de schémas lisses et propres, nous expliquons comment DAG éclaire la théorie classique de la déformation. Dans les deux dernières sections, nous introduisons les formes différentielles sur le champs dérivées, puis nous nous spécialisons au formes symplectiques décalées, donnant les principaux théorèmes d'existence prouvés dans [17].

Traces non-linéaires
David Ben-Zvi & David Nadler 39

Nous combinons la théorie des traces en algèbre homotopique avec la théorie des faisceaux en géométrie algébrique dérivée afin déduire des formules de caractéres et de points fixes. La notion de dimension (ou homologie de Hochschild) d'un objet dualisable dans le contexte de l'algèbre supérieure fournit un cadre unificateur pour les notions classiques telles que les caractéristiques d'Euler, les caractères de Chern et les caractères des représentations de groupes. De plus, la fonctorialité de ces dimensions clarifient certaines formules célèbres et les étendent à de nouveaux contextes.

Nous observons qu'il est avantageux de calculer les dimensions, les traces et leurs fonctorialités directement dans le cadre géométrique non-linéaire des catégories de correspondances, où elles sont respectivement identifiés directement avec (les versions dérivées des) les espaces de lacets, les lieux de points fixes et leurs fonctorialités. Il en résulte des versions universelles non-linéaires des formules de traces de Grothendieck-Riemann-Roch, de la formule de Atiyah-Bott-Lefschetz et des formules de caractères de Frobenius-Weyl. Il est par ailleurs possible de linéariser en appliquant certaines théories de faisceaux, telle que la théorie des faisceaux ind-cohérents et des D-modules construites par Gaitsgory-Rozenblyum [16]. Cela permet de retrouver les formules classiques, valables en familles et sans hypothèse de lissité ou de transversalité. D'un autre côté, notre formalisme s'applique également à certains invariants catégoriques supérieurs non présents dans le cadre linéaire, tels que la notion de caractères des actions de groupes sur des catégories.

Problèmes de modules formels et champs dérivés formels
Damien Calaque & Julien Grivaux 85

Cet article présente un survol des problèmes de modules formels. Nous commençons par une introduction aux problèmes de modules formels pointés, et esquissons la démonstration d'un théorème (dû à Lurie et Pridham) donnant une formulation mathématique précise à la philosophie dite de « déformation dérivée » de Drinfeld. Ce résultat donne une correspondance entre les problèmes de modules formels et les algèbres de Lie différentielles graduées. Dans un second temps, nous présentons la théorie générale des contextes de déformation de Lurie, en insistant sur la notion (plus symétrique) de contexte de dualité de Koszul. Nous appliquons ensuite ce cadre général au cas des problèmes de modules formels non scindés sous un schéma affine dérivé fixé ; cette situation a été étudiée récemment par Nuiten, et nécessite de remplacer les algèbres de Lie différentielle graduée par des a lgébroïdes de Lie différentiels gradués. Dans la dernière partie, nous esquissons la globalisation au cas plus général des épaississements formels de champs dérivés, et suggérons une approche alternative à certains résultats de Gaitsgory et Rozenblyum.

Théorie de Gromov-Witten à l'aide de la géométrie algébrique dérivée
Etienne Mann & Marco Robalo 147

Dans cet article de synthèse nous ajoutons deux nouveaux résultats qui n'étaient pas dans notre papier [42]. En utilisant l'action des membranes découvertes par Bertrand Toën, nous construisons une action associative relachée de l'opérade des courbes stables de genre zéro sur une variété projective lisse, vue comme une correspondance dans les champs dérivés. Cette action encode la théorie de Gromov-Witten en termes purement géométrique.

Conférences sur la géométrie de Poisson décalée
Pavel Safronov 187

Nous introduisons la théorie de structures de Poisson décalées (supérieures) sur les champs algébriques dérivés. Nous discutons de plusieurs résultats fondamentaux, tels que les intersections coisotropes, l'additivité de Poisson, et une comparaison avec la théorie des structures symplectiques décalées. Enfin, nous fournissons plusieurs exemples de structures de Poisson décalées liées aux structures de Poisson-Lie et aux espaces de modules de fibrés.

ABSTRACTS

Introductory topics in derived algebraic geometry
TONY PANTEV & GABRIELE VEZZOSI 1

We give a quick introduction to derived algebraic geometry (DAG) sampling basic constructions and techniques. We discuss affine derived schemes, derived algebraic stacks, and the Artin-Lurie representability theorem. Through the example of deformations of smooth and proper schemes, we explain how DAG sheds light on classical deformation theory. In the last two sections, we introduce differential forms on derived stacks, and then specialize to shifted symplectic forms, giving the main existence theorems proved in [17].

Nonlinear Traces
DAVID BEN-ZVI & DAVID NADLER 39

We combine the theory of traces in homotopical algebra with sheaf theory in derived algebraic geometry to deduce general fixed point and character formulas. The formalism of dimension (or Hochschild homology) of a dualizable object in the context of higher algebra provides a unifying framework for classical notions such as Euler characteristics, Chern characters, and characters of group representations. Moreover, the simple functoriality properties of dimensions clarify celebrated identities and extend them to new contexts.

We observe that it is advantageous to calculate dimensions, traces and their functoriality directly in the nonlinear geometric setting of correspondence categories, where they are directly identified with (derived versions of) loop spaces, fixed point loci and loop maps, respectively. This results in universal nonlinear versions of Grothendieck-Riemann-Roch theorems, Atiyah-Bott-Lefschetz trace formulas, and Frobenius-Weyl character formulas. We can then linearize by applying sheaf theories, such as the theories of ind-coherent sheaves and $\mathscr{D}$-modules constructed by Gaitsgory-Rozenblyum [16]. This recovers the familiar classical identities, in families and without any smoothness or transversality assumptions. On the other hand, the formalism also applies to higher categorical settings not captured within a linear framework, such as characters of group actions on categories.

Formal moduli problems and formal derived stacks
DAMIEN CALAQUE & JULIEN GRIVAUX 85

This paper presents a survey on formal moduli problems. It starts with an introduction to pointed formal moduli problems and a sketch of proof of a Theorem (independently proven by Lurie and Pridham) which gives a precise mathematical formulation for Drinfeld's *derived deformation theory* philosophy. This theorem provides a correspondence between formal moduli problems and differential graded Lie algebras. The second part deals with Lurie's general theory of deformation contexts, which we present in a slightly different way than the original paper, emphasizing the (more symmetric) notion of Koszul duality contexts and morphisms thereof. In the third part, we explain how to apply this machinery to the case of non-split formal moduli problems under a given derived affine scheme; this situation has been dealt with recently by Joost Nuiten, and requires to replace differential graded Lie algebras with differential graded Lie algebroids. In the last part, we globalize this to the more general setting of formal thickenings of derived stacks, and suggest an alternative approach to results of Gaitsgory and Rozenblyum.

Gromov-Witten theory with derived algebraic geometry
ETIENNE MANN & MARCO ROBALO 147

In this survey we add two new results that are not in our paper [42]. Using the idea of brane actions discovered by Toën, we construct a lax associative action of the operad of stable curves of genus zero on a smooth variety X seen as an object in correspondences in derived stacks. This action encodes the Gromov-Witten theory of X in purely geometrical terms.

Lectures on shifted Poisson geometry
PAVEL SAFRONOV 187

These are expanded notes from lectures given at the États de la Recherche workshop on "Derived algebraic geometry and interactions". These notes serve as an introduction to the emerging theory of Poisson structures on derived stacks.

INTRODUCTION

Derived algebraic geometry, often refer to as "DAG," was formally born in the early 2000s. It proposes a general setting for algebraic geometry particularly well suited to deal with non-generic situations such as non-transversal intersections and non-free group actions. We refer the reader to [**13**] for an extended historical introduction to the subject, and we restrict ourselves to a shorter overview below.

The origin of the main ideas that has led to the present shape of DAG can be tracked back to the 1950s and to the so-called intersection formula of Serre of [**12**]. In this work, intersection numbers of two subvarieties, in an ambient smooth variety, are defined in terms of the alternating length of the higher Tor between the corresponding structure sheaves. Today, this is interpreted as the length, at the generic points, of the structure sheaf of the derived intersection of the two varieties, which is a special example of a *derived scheme*. Serre's intersection formula is probably the very first instance of a derived scheme used in the context of algebraic geometry and is often referred to as the starting point of the whole subject.

From a different perspective, another source of ideas came from deformation theory with the derived deformation theory program (DDT for short) initiated by Drinfeld in a letter to Schechtmann (see [**6**]). This program has been developed further by various authors and can be subsumed by the fact that any reasonable deformation theory problem is governed by a differential graded Lie algebra (*dg-Lie algebras* for short). When a deformation problem X is associated to a dg-Lie algebra $\mathfrak{g}$, the geometric invariants associated to X can be extracted by algebraic construction out of $\mathfrak{g}$. For instance, the ring of formal functions on X is given by the 0-th cohomology group $H^0(\mathfrak{g}, k)$ of $\mathfrak{g}$ with coefficients in the trivial module k, or equivalently $H^0(C^*(\mathfrak{g}))$ where $C^*(\mathfrak{g})$ is the Eilenberg-MacLane complex of the dg-Lie algebra $\mathfrak{g}$. It turns out that the other cohomologies $H^i(\mathfrak{g}, k)$ are in general non-zero for $i > 0$, and these groups can be interpreted as coherent sheaves on X which are sometimes called the *virtual structure sheaves on* X. When X exists as a global moduli space, the typical example here is the moduli of stable maps of Kontsevich, these sheaves exist globally as coherent sheaves on X and encode the so-called *virtual fundamental class* of X, in a very similar fashion that the higher Tor's of Serre's intersection formula encode the correct intersection numbers.

Both of these streams of ideas reminded above have been eventually formalized by Ciocan-Fontanine and Kapranov into a notion of *dg-scheme* (see **[5]**), which is a first approximation of what a *derived scheme* is. The formalism of dg-schemes has been very useful to prove various results, such as the construction of virtual classes, the virtual Riemann-Roch formula, the existence of dg-enhancements of Hilbert and Quot schemes and so on and so forth. However, the theory of dg-schemes has been recognized too rigid by many aspects, and in particular the theory was missing a reasonable *functor of point* view point, making it difficult to consider representability theorems, or the notion of stacks as well as more complicated objects such as higher stacks.

The modern foundations of DAG are based on the more flexible notion of *derived schemes* and of *derived stacks*, far-reaching generalizations of schemes and stacks. The reader will find all the precise definitions in the first contribution of this volume, as well as in **[9, 13, 15]**. Let us remind here that a derived scheme is a simple data consisting of a space X together with a sheaf of commutative dg-algebras $\mathcal{O}_X$ (or more generally simplicial commutative rings if one works outside of characteristic zero), satisfying the following two simple conditions.

(1) The ringed space $(X, H^0(\mathcal{O}_X))$ is a scheme.

(2) The sheaves $H^i(\mathcal{O}_X)$ are quasi-coherent sheaves on $(X, H^0(\mathcal{O}_X))$.

For a derived scheme $(X, \mathcal{O}_X)$, the scheme $(X, H^0(\mathcal{O}_X))$ is called the *truncation* or the *classical part* of the derived schemes. The sheaves $H^i(\mathcal{O}_X)$ are themselves the virtual structure sheaves mentioned above, and they combine in the *derived structure sheaf* namely $\mathcal{O}_X$ itself. Even though derived schemes are perfectly easy to define, it turns out that defining the notion of morphisms between derived schemes requires some rather involved categorical constructions. The reason is that the sheaf $\mathcal{O}_X$ of commutative dg-algebras must be considered only up to quasi-isomorphism, and the correct manner to deal with this is to use higher category theory, or its avatar Quillen homotopical algebra. Unfortunately, the use of higher categories or model categories is totally unavoidable in order to get a flexible theory of derived schemes for which for instance gluing (also called descent) is possible. Fortunately, this work has been done some years ago and is the content of the foundational texts on DAG **[9]** and **[15]**.

DAG is today a well funded subject, and is used in many domains of geometry. To mention a few: the geometric Langlands correspondence **[7]**, geometric representation theory **[1]**, quantization of symplectic and Poisson structures **[10, 4, 11]**, enumerative geometry **[3]**, and more recently p-adic geometry **[2]**. We would like here to insist on two particular directions in which DAG has found applications recently, categorification and formal/infinitesimal geometry, which are the main themes of the contributions to the present volume. On the categorification side, it is important to note that the derived category of a derived scheme is most often better behaved than the derived category of its truncation, this is particularly true when considering moduli spaces. One reason for this is that the base change formula is always satisfied in the setting of derived schemes and derived stacks, making the action of natural correspondences on derived categories of certain moduli spaces more regular and less pathological than the induced action on their truncations. This principle has led to many interesting applications of DAG to categorification. In another direction, the infinitesimal and formal study of derived schemes and stacks is also more natural and easier to compute explicitly. The reason here can be already seen at the level of tangent complexes. The tangent complexes of classical (underived) moduli spaces are often impossible to compute explicitly, as these are most of the time unbounded. However, their derived counter-parts have easy modular interpretations in terms of cohomology and thus have explicit descriptions. This has been used recently in order to study the infinitesimal and formal structures of derived schemes and stacks, and has made possible for instance the theory of shifted symplectic and Poisson structures. In the same vein, derived stacks and formal derived stacks have been used in order to integrate dg-Lie algebroids, the same manner than Lie groupoids and differential stacks were used in order to integrate Lie algebroids (see [**16**]).

The present volume covers the content of one of the session of "États de la recherche" on derived algebraic geometry which has been held in Toulouse in June 2017. The purpose of this event was to gather world experts who are using DAG to approach different problems in algebraic geometry, but also in Lie theory and geometric representation theory. The week was organized in several mini-courses by David Ben-Zvi, Damien Calaque, Dominic Joyce, Etienne Mann, Tony Pantev, Marco Robalo, Pavel Safronov and Gabriele Vezzosi, from which the contributions to this volume are taken.

Introductory topics in derived algebraic geometry, by Tony Pantev and Gabriele Vezzosi, is a short introduction to the main ideas and concepts of DAG and its applications to the notions of shifted symplectic and Poisson structures. It covers the basic definitions of derived schemes and derived stacks, starting from scratch with the homotopy theory of commutative dg-algberas, cotangent complexes etc. It covers a wide range of fundamental results such as Lurie's representability theorem, the interpretation of derived deformation theory, and the notion of differential forms and symplectic structures in the derived setting. This contribution serves as an expanded introduction to the subject for the reader who is not familiar to DAG, and contains the details of all the notions we have discussed in this introduction.

In *Nonlinear traces*, by David Ben-Zvi and David Nadler, the authors present a general categorical formalism for traces, with applications to algebraic geometry and Grothendieck-Riemann-Roch's type formula. One of the leading idea is that the derived category of varieties, or more generally of certain stacks, behaves like finite-dimensional vector spaces when suitably considered as dg-categories or stable ∞-categories. It is then possible to define the traces of various endomorphisms, including the identity morphism whose trace is called the *dimension*. By observation, the traces are preserved by symmetric monoidal functors, and when applied to derived categories of sheaves (quasi-coherent, ind-coherent or D-modules) this preservation produces formula of Grothendieck-Riemann-Roch type. The authors present several very nice applications, such as Atiyah-Bott-Lefschetz fixed point formula. In this setting, DAG is used in an essential way to insure that correspondence acts functorially on various derived categories, in a compatible manner with compositions. For this to be true, composition of correspondences, which are given by certain fiber products, must be computed in the category of derived schemes and derived stacks.

The paper *Formal moduli problems and formal derived stacks*, by Damien Calaque and Julien Grivaux, contains a fundational work on deformation theory. It is a well-established statement (see **[14]**) that formal moduli problems over a field are classified by dg-Lie algebras. The extension of this important classification result to more general bases is an important question which is addressed by the authors. They study the notion of formal moduli problems over and under a base derived scheme X. These can be considered as families of formal moduli problems parametrized by X in some sense. The main result states that these are classified by dg-Lie algebras over X, and dg-Lie algebroids over X, depending of the fact that one considers split (i.e. pointed) or non-split formal moduli problems. This result can be understood as an integration result, making a correspondence between dg-Lie algebroids and certain derived formal groupoids over X, similar to the well-known Lie correspondence between Lie algebroids and Lie groupoids (see **[16]**). These results are for instance useful for the study of the formal completions of a scheme X along a closed subscheme Y, by means of a dg-Lie algebroid over Y.

The subject *Gromov-Witten theory with derived algebraic geometry*, by Etienne Mann and Marco Robalo, is a construction of a categorification of the Gromov-Witten invariants. The Gromov-Witten invariants of a variety X can be described by an action of the modular operad on the cohomology of X. In this work the authors categorify this action by showing that it is induced by an action of the family of derived categories $D(\bar{\mathcal{M}}_{g,n})$ on the derived category $D(X)$. For this, it is shown that the family of categories $D(\bar{\mathcal{M}}_{g,n})$ forms a *categorical operad.* An action of this categorical operad on $D(X)$ is constructed using the derived moduli space of stable maps $\mathbb{R}\bar{\mathcal{M}}_{g,n}(X)$, considered as a correspondence

$$\bar{\mathcal{M}}_{g,n+1} \times X \longleftarrow \mathbb{R}\bar{\mathcal{M}}_{g,n+1}(X) \longrightarrow X^n.$$

The fact that this action satisfies associativity is the core of the main result, and is proven by using the base change formula in DAG as well as the theory of ∞-operads of **[8]**. The more traditional Gromov-Witten invariants are then recovered by passing to K-theory or periodic cyclic homology.

In *Lectures on shifted Poisson geometry*, by Pavel Safronov, applications and interactions between DAG and geometric representation theory are presented. The main subject is that of *shifted Poisson structures* and their quantizations. Shifted Poisson structures are Poisson brackets defined on functions of a derived scheme or a derived stack, with some possibly non-trivial cohomological degree. In the classical setting of smooth varieties they are the usual Poisson structures, but there are lots of natural examples with non-zero cohomological shift, the most fundamental being degree 2 Poisson structures on the classifying stack BG of a reductive group G. In this manuscript the author explains how to recover all the standard notions appearing in quantum groups using shifted Poisson structures, starting from this fundamental example of BG. He also presents the comparisons between shifted Poisson structures and the Lagrangian morphisms, a far-reaching generalization of the well known relation between Poisson bracket and symplectic groupoid actions. The paper ends with some application to the FeiginOdesskii Poisson structure on the moduli space of bundles on an elliptic curve as well as the construction of the Poisson structure on the moduli space of monopoles.

As a final note, we would like to point out that the lectures and the slides are all available at the following address:

`https://www.math.univ-toulouse.fr/dagit/index.php?page=video.php`

Acknowledgments. – We would like to warmly thank all the participants to the session of États de la recherche *Derived Algebraic Geometry and Interactions*, that was held in Toulouse 12-15 June 2017. A particular thank to all the lecturers and to their contributions to this volume, that, we hope, will be useful to students and colleagues who would like to learn the subject.

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Bertrand Toën and Michel Vaquié
Toulouse, November 2019.

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