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CHARACTERISTIC CYCLES AND THE MICROLOCAL GEOMETRY OF THE GAUSS MAP, I

BY Thomas KRÄMER

ABSTRACT. – We propose two new approaches to the Tannakian groups of holonomic $\mathscr{D}$-modules on abelian varieties. The first is an interpretation in terms of principal bundles given by the Fourier-Mukai transform, which shows that they are almost connected. The second constructs a microlocalization functor relating characteristic cycles to Weyl group orbits of weights. This explains the ubiquity of minuscule representations, and we illustrate it with a Torelli theorem and with a bound for decompositions of a given subvariety as a sum of subvarieties. The appendix sketches a twistor variant that may be useful for $\mathscr{D}$-modules not coming from Hodge theory.

RÉSUMÉ. – Nous proposons deux nouvelles approches aux groupes tannakiens des $\mathscr{D}$-modules holonomes sur les variétés abéliennes. La première est une interprétation en termes de fibrés principaux définis par la transformation de Fourier-Mukai, ce qui implique qu'ils sont essentiellement connexes. La deuxième fournit un foncteur de microlocalisation qui relie les cycles caractéristiques aux orbites des groupes de Weyl sur les poids. Cela explique l'ubiquité des représentations minuscules, et nous l'illustrons par un théorème de Torelli et par une borne pour les décompositions d'une sous-variété donnée comme somme d'autres sous-variétés. L'appendice donne une variante twistorielle qui peut être utile pour les $\mathscr{D}$-modules ne provenant pas de la théorie de Hodge.

1. Introduction

A common thread in algebraic geometry is the study of various realizations of the six functor formalism like holonomic $\mathscr{D}$-modules, Hodge modules or ℓ-adic perverse sheaves. On semiabelian varieties one has a Tannakian correspondence between such objects and representations of certain algebraic groups [31, 26, 16]. These groups are interesting not only over finite fields where they are related to equidistribution [25] and generic vanishing [52], but also over the complex numbers where they can be used for instance to study singular subvarieties [29, 27]. Our goal in the present paper and its sequel [28] is to understand the geometric meaning of the Tannakian formalism in the case of holonomic $\mathscr{D}$-modules on complex abelian varieties.

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1.a. Motivation and overview

Let A be a complex abelian variety. As we will recall in the next section, the Tannakian formalism naturally attaches to any holonomic $\mathscr{D}_A$-module $\mathscr{M}$ an affine complex algebraic group $G(\mathscr{M})$; this paper is motivated by the following

EXAMPLE 1.1. – For a proper closed subvariety $Z \subset A$, let $\mathscr{M} = \delta_Z$ be the regular holonomic $\mathscr{D}_A$-module whose associated perverse sheaf is the perverse intersection complex of the subvariety. There are two extreme cases:

(a) If Z is a finite set of points, one easily sees that $G(\delta_Z) = \mathrm{Hom}(\Gamma, \mathbb{G}_m)$ is the Cartier dual of the group $\Gamma \subset A(\mathbb{C})$ generated by these points.

(b) If Z is a divisor on A, then the group $G(\delta_Z)$ is related to the Gauss map sending a smooth point of the divisor to its normal direction. An interesting case is the theta divisor on intermediate Jacobians of cubic threefolds, for which $G(\delta_Z)$ is an exceptional group of type E_6 and the monodromy of the Gauss map of the theta divisor can be identified with the Weyl group $W(E_6)$. See [27].

We generalize both observations to the groups $G = G(\mathscr{M})$ for any $\mathscr{M}$. Our first approach identifies these groups as structure groups of principal bundles by Schnell's work on the Fourier-Mukai transform [49]. If $G^\circ \subseteq G$ denotes the connected component of the identity, we deduce that

$$G/G^\circ = \mathrm{Hom}(X, \mathbb{G}_m) \text{ for a finite subgroup } X \subset A(\mathbb{C})$$

as in the first example above (Theorem 1.5). This complements the analogous theorem by Weissauer for ℓ-adic perverse sheaves on abelian varieties over finitely generated fields [55, Th. 2 and the paragraph after Lemma 11]. The latter works over base fields of arbitrary characteristic, but since it is a result about perverse sheaves, it can in characteristic zero only deal with *regular* holonomic $\mathscr{D}$-modules. Our proof is very different and works directly with $\mathscr{D}$-modules—as such it is restricted to base fields of characteristic zero but includes the case of *irregular* holonomic $\mathscr{D}$-modules.

Given the above result, we have a fairly good understanding of the group G/G° of connected components, so that the main task will be to understand the connected component $G^\circ \subseteq G$ of the identity for the groups $G = G(\mathscr{M})$. If $\mathscr{M}$ is semisimple, then this is a connected reductive group; our second approach studies its root system via subgroups of multiplicative type that are related to Gauss maps. For this we consider the characteristic variety $\mathrm{Char}(\mathscr{M}) \subset T^*A$, which is a conic Lagrangian subvariety of the cotangent bundle. For abelian varieties the cotangent bundle is the trivial bundle $T^*A = A \times V$ with fiber $V = H^0(A, \Omega^1_A)$. We define the *Gauss map* to be the projection

$$\gamma : \mathrm{Char}(\mathscr{M}) \subset T^*A = A \times V \twoheadrightarrow V$$

to the cotangent space. This definition generalizes the classical notion of Gauss maps for divisors on abelian varieties. In the cases relevant to this paper, the map γ is dominant and generically finite. In Sections 4 and 5 we construct an embedding $\mathrm{Hom}(\Gamma, \mathbb{G}_m) \hookrightarrow G$ for the subgroup

$$\Gamma = \langle\, a \in A(\mathbb{C}) \mid (a, u) \in \mathrm{Char}(\mathscr{M}) \,\rangle \subset A(\mathbb{C})$$

that is generated by the finitely many points in the fiber of the Gauss map over a fixed, very general $u \in V(\mathbb{C})$ (by a *very general* point on a variety we mean a point outside countably many proper closed subvarieties). The resulting link between Weyl groups and Gauss maps generalizes the second example from above (Corollary 1.8), explains the ubiquity of minuscule representations, and provides a way to compute Tannaka groups that covers all previously known cases in a uniform way (Theorem 2.1).

For geometric applications we refer the reader to [28] and only illustrate our ideas with two simple examples: A lower bound for summands of subvarieties (Proposition 2.3) that shows why for intermediate Jacobians of cubic threefolds the theta divisor is not a sum of curves, a special case of Schreieder's beautiful result [50]; and a Torelli theorem which generalizes those for curves and Fano surfaces of cubic threefolds by recovering a smooth subvariety from Tannakian data (Corollary 2.5). The appendix gives an alternative approach via the theory of twistor modules by Mochizuki and Sabbah that may be useful for holonomic $\mathscr{D}$-modules not coming from Hodge modules.

1.b. The Tannakian framework

Let $\mathscr{D}_A$ be the sheaf of algebraic differential operators and

$$\mathrm{Hol}(\mathscr{D}_A) \subset \mathrm{D}^b_{\mathrm{hol}}(\mathscr{D}_A)$$

the abelian category of holonomic right $\mathscr{D}_A$-modules resp. the derived category of bounded algebraic $\mathscr{D}_A$-module complexes with holonomic cohomology sheaves. On the latter the addition morphism $a : A \times A \to A, (x, y) \mapsto x + y$ defines a convolution product

$$\mathscr{M}_1 * \mathscr{M}_2 = a_\dagger(\mathscr{M}_1 \boxtimes \mathscr{M}_2)$$

which makes the derived category into a *tensor category*, i.e., a $\mathbb{C}$-linear symmetric monoidal category. The abelian subcategory $\mathrm{Hol}(\mathscr{D}_A) \subset \mathrm{D}^b_{\mathrm{hol}}(\mathscr{D}_A)$ is not stable under convolution, but this can be overcome by passing to a quotient category as described in a more general axiomatic framework in [31, Th. 13.2]. For convenience we briefly recall how the construction works in the case relevant for this paper: For any $\mathscr{M} \in \mathrm{Hol}(\mathscr{D}_A)$ the analytic de Rham complex

$$\mathrm{DR}(\mathscr{M}) = \Big[\underbrace{\cdots \longrightarrow \mathscr{M} \otimes_{\mathscr{O}_A} \wedge^2 \mathscr{T}_A^{\mathrm{an}}}_{\text{degree } -2} \longrightarrow \underbrace{\mathscr{M} \otimes_{\mathscr{O}_A} \mathscr{T}_A^{\mathrm{an}}}_{\text{degree } -1} \longrightarrow \underbrace{\mathscr{M} \otimes_{\mathscr{O}_A} \mathscr{O}_A^{\mathrm{an}}}_{\text{degree } 0}\Big]$$

is a perverse sheaf whose hypercohomology has non-negative Euler characteristic by Kashiwara's index formula and the fact that the characteristic cycle of any perverse sheaf is effective [15, Cor. 1.4]. In what follows we say that $\mathscr{M}$ is *negligible* if the Euler characteristic of its hypercohomology vanishes. Since the Euler characteristic is additive on short exact sequences, the negligible modules form a thick abelian subcategory $\mathrm{S}(A) \subset \mathrm{Hol}(\mathscr{D}_A)$. Furthermore, if $\mathrm{T}(A) \subset \mathrm{D}^b_{\mathrm{hol}}(\mathscr{D}_A)$ denotes the thick triangulated subcategory of complexes whose cohomology sheaves are negligible, then by [16, Prop. 3.6.1] the abelian quotient category

$$\mathrm{M}(A) = \mathrm{Hol}(\mathscr{D}_A)/\mathrm{S}(A)$$